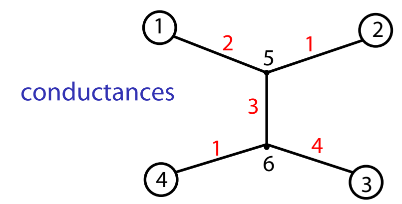
Response Matrix M : $R^4 \longrightarrow R^4$ from [1]

voltages applied at leaves — currents in (+) or out (-) at each leaf

Note: L and -L are often interchanged



$$-L=\begin{array}{c} \quad A \qquad\qquad\qquad\qquad B \\ \begin{pmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 & 0 & 4 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 2 & 1 & 0 & 0 & -6 & 3 \\ 0 & 0 & 4 & 1 & 3 & -8 \end{pmatrix} \\ \qquad\qquad\qquad\qquad\qquad C \end{array}$$

$M = A-BC^{-1}B^T$

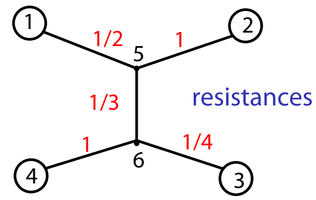
$$M = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -6 & 3 \\ 3 & -8 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \end{pmatrix}$$

$$= \frac{1}{39}\begin{pmatrix} -46 & 16 & 24 & 6 \\ 16 & -31 & 12 & 3 \\ 24 & 12 & -60 & 24 \\ 6 & 3 & 24 & -33 \end{pmatrix} = \begin{pmatrix} -\frac{46}{39} & \frac{16}{39} & \frac{8}{13} & \frac{2}{13} \\ \frac{16}{39} & -\frac{31}{39} & \frac{4}{13} & \frac{1}{13} \\ \frac{8}{13} & \frac{4}{13} & -\frac{20}{13} & \frac{8}{13} \\ \frac{2}{13} & \frac{1}{13} & \frac{8}{13} & -\frac{11}{13} \end{pmatrix}$$

Resistance Matrix W

$V = IR,\ C = 1/R$

metric of resistance distances



$L+(1/6)J =$ where J is matrix of all 1's

$$\begin{pmatrix} \frac{13}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{11}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{7}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{25}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{23}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{7}{6} & \frac{1}{6} & -\frac{5}{6} \\ -\frac{11}{6} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} & \frac{37}{6} & -\frac{17}{6} \\ \frac{1}{6} & \frac{1}{6} & -\frac{23}{6} & -\frac{5}{6} & -\frac{17}{6} & \frac{49}{6} \end{pmatrix}$$

$$X = (L+(1/6)J)^{-1} = \frac{1}{144}\begin{pmatrix} 95 & 11 & 5 & -13 & 35 & 11 \\ 11 & 143 & -7 & -25 & 23 & -1 \\ 5 & -7 & 71 & 17 & 17 & 41 \\ -13 & -25 & 17 & 143 & -1 & 23 \\ 35 & 23 & 17 & -1 & 47 & 23 \\ 11 & -1 & 41 & 23 & 23 & 47 \end{pmatrix}$$

Pseudoinverse: $L^+ =$

$$\frac{1}{144}\begin{pmatrix} 71 & -13 & -19 & -37 & 11 & -13 \\ -13 & 119 & -31 & -49 & -1 & -25 \\ -19 & -31 & 47 & -7 & -7 & 17 \\ -37 & -49 & -7 & 119 & -25 & -1 \\ 11 & -1 & -7 & -25 & 23 & -1 \\ -13 & -25 & 17 & -1 & -1 & 23 \end{pmatrix}$$

$W = X_d J + J X_d - 2X$

$$= (1/144)\left(\begin{pmatrix} 95 & 0 & 0 & 0 \\ 0 & 143 & 0 & 0 \\ 0 & 0 & 71 & 0 \\ 0 & 0 & 0 & 143 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} 95 & 0 & 0 & 0 \\ 0 & 143 & 0 & 0 \\ 0 & 0 & 71 & 0 \\ 0 & 0 & 0 & 143 \end{pmatrix} - 2\begin{pmatrix} 95 & 11 & 5 & -13 \\ 11 & 143 & -7 & -25 \\ 5 & -7 & 71 & 17 \\ -13 & -25 & 17 & 143 \end{pmatrix} \right)$$

$$= (1/144)\begin{pmatrix} 0 & 216 & 156 & 264 \\ 216 & 0 & 228 & 336 \\ 156 & 228 & 0 & 180 \\ 264 & 336 & 180 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{3}{2} & \frac{13}{12} & \frac{11}{6} \\ \frac{3}{2} & 0 & \frac{19}{12} & \frac{7}{3} \\ \frac{13}{12} & \frac{19}{12} & 0 & \frac{5}{4} \\ \frac{11}{6} & \frac{7}{3} & \frac{5}{4} & 0 \end{pmatrix}$$

Note :the same calculation for W works with X or with pseudoinverse of L.

Quest. 1 Show that W is Kalmanson.
Quest. 2 Find a function from W to M.
Quest. 3 Relate the complex of circular split networks to the complex of circular planar networks

[1] J. Alman, C. Lian, B. Tran, slides: Response Matrices of Circular Planar Electrical Networks

$$Z = (-M)^+ = -\frac{1}{192}\begin{pmatrix} -97 & 23 & 19 & 55 \\ 23 & -145 & 43 & 79 \\ 19 & 43 & -73 & 11 \\ 55 & 79 & 11 & -145 \end{pmatrix}$$

$W = Z_d J + J Z_d - 2Z$ example: $(97+145+2(23))/192 = 3/2$

$$= (-1/192)\left(\begin{pmatrix} -97 & 0 & 0 & 0 \\ 0 & -145 & 0 & 0 \\ 0 & 0 & -73 & 0 \\ 0 & 0 & 0 & -145 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} -97 & 0 & 0 & 0 \\ 0 & -145 & 0 & 0 \\ 0 & 0 & -73 & 0 \\ 0 & 0 & 0 & -145 \end{pmatrix} -2 \begin{pmatrix} -97 & 23 & 19 & 55 \\ 23 & -145 & 43 & 79 \\ 19 & 43 & -73 & 11 \\ 55 & 79 & 11 & -145 \end{pmatrix} \right)$$

Interpretation:

M(i,j) is the current at leaf j when voltage of 1 (positive terminal) is applied to leaf i, and voltage 0 (negative terminal) to all other leaves simultaneously.

W(i,j) is the resistance between leaf i and leaf j; so equals 1/(current) at leaf i (and j) when voltage 1 (positive terminal) is applied to leaf i and voltage 0 (negative terminal) is applied at leaf j, (or vice versa) and other leaves are not connected to the circuit!