Project Summary: Geometric combinatorial Hopf algebras and modules.

Greater understanding leads to greater utility. Electric lighting existed prior to the theory of quantum electrodynamics. The theory, however, certainly extends our understanding of electrons and photons—and now we can take images of the magnetic resonances in the brain of a cancer patient. Successful calculations using the techniques of renormalization were performed for nearly half a century without the help of Hopf algebras. The discovery of Hopf algebras underlying renormalization, however, allowed researchers to put renormalization on a firm mathematical footing—and enables current application to other quantum field theories, including quantum gravity. In this same spirit we are convinced that greater understanding of the crucial mathematical ingredients of the Connes-Kreimer algebra, as well as its place in a larger family of structures, will be extremely valuable. The goal of this proposed project is to understand ways in which Hopf structures are predicted by the geometric combinatorics of graphs and their subgraphs. Specifically we plan to study a large family of generalizations of both the Loday-Ronco and the Malvenuto-Reutanauer Hopf algebras. Our new algebras arise both from interesting sequences of Devadoss's graph-associahedra and from new combinatorial polytopes. In addition, we plan to carry this program further to the study of Hopf modules arising from the graph-multiplihedra and their quotients. A third, parallel stage consists of investigating the applicability of our discoveries to the geometric combinatorics of cluster algebras on one hand and nested complexes on the other.

Renormalization refers to a family of algorithms that generate counterterms to deal with divergences arising from loops in the Feynman diagrams. Divergences can be nested, disjoint or overlapping, but the overlapping divergences can be resolved into nested or disjoint. The operations of product, coproduct and antipode in the Hopf algebra of Feynman diagrams rely on insertion of and restriction to subdiagrams. The residue after extraction of a subgraph also appears in the formula as a reconnected complement. Here is the formula for coproduct applied to a Feynman diagram Γ with subdiagrams γ, including the empty and trivial subdiagrams:

$$\Delta\Gamma = \sum_{\gamma\subset\Gamma} \gamma \otimes \Gamma/\gamma$$

Here Γ/γ is the diagram achieved by removing γ and then reconnecting the vertices of the graph that were connected through γ.

We have noticed analogous features in geometric combinatorics. For example, in the theory of graph-associahedra, given a graph G on $n+1$ nodes an n-dimensional convex polytope K_G is constructed. Lower dimensional faces correspond to collections of subgraphs for which each pair is either nested or disjoint. The geometry of faces is described by restricting the polytope construction to subgraphs and to their reconnected complements. The formula for the facet of a graph-associahedron corresponding to a connected subgraph t of a graph G is $K_t \times K_{G/t}$ where G/t is again the reconnected complement. Each facet corresponds to a connected subgraph, so we can write:

$$K_G = \bigcup_{t\subset G} K_t \times K_{G/t}.$$

The situation is more complicated than suggested by a simple analogy. Sequences of polytopes such as the permutohedra and the associahedra (which both underlie important Hopf algebras) enjoy special recursive properties. The first portion of our plan will involve extension of those properties in various directions: to graph associahedra, Postnikov's generalized permutohedra, and to Fomin and Zelevinsky's generalized associahedra. The second broad area of inquiry will be in colored versions of the combinatorial objects, the multiplihedral polytopes they form, and the Hopf modules they underlie.