Project description: Enrichment and its relationship to classifying spaces

1. Introduction

There is an ongoing massive effort by many researchers to link category theory and geometry, especially homotopy coherence and categorical coherence. This effort has as a partial goal that of understanding the categories and functors that correspond to loop spaces and their associated topological functors. This proposal pursues the hints of a categorical delooping that are suggested when enrichment is iterated. At each stage of successive enrichments, the number of monoidal products is decremented and the categorical dimension is incremented. This is mirrored by topology. When we consider the loop space of a topological space, we see that paths (or 1–cells) in the original are now points (or objects) in the derived space. There is also automatically a product structure on the points in the derived space, where multiplication is given by concatenation of loops. Delooping is the inverse functor here, and thus involves shifting objects to the status of 1–cells and decreasing the number of ways to multiply. It has been shown that, quite analogously, for $\mathcal{V}$ k–fold monoidal the structure of a $(k-n)$–fold monoidal strict $(n+1)$–category is possessed by $\mathcal{V}$–n–Cat [16], [17].

An analogy between morphological processes is usually codified as a natural transformation between functors. In this case we will be looking for higher natural transformations, or weakened versions of natural transformations, between a variety of enrichment functors on one hand and corresponding sorts of delooping functors of loop spaces on the other. Instances of the transformation will be generalizations of the nerve functor on 1-categories, which preserves homotopy equivalence. In [50] Street defines the nerve of a strict n-category. Recently Duskin in [14] has worked out the description of the nerve of a bicategory. A second part of the latter paper promises the full description of the functor including how it takes morphisms of bicategories to continuous maps. The available choices of categorical models of loop spaces and the various sorts of enrichment divide the work into several areas; roughly delineated by a choice between single and plural categorical dimensions and a choice between weak and strict versions of enrichment.

Two basically new ideas are to be exploited in this research. Briefly they are: 1) using operad modules to characterize enrichment, and 2) using stacking products in categories of decorated n-dimensional Young diagram to model loop spaces. Classically the method for weakening enrichment has been to talk of composition maps which are given by the action of an operad $\mathcal{C}$ on products of morphism objects $\mathcal{A}(A,B)$:

$$\mathcal{C}(n)\otimes\mathcal{A}(A_{n-1},A_n)\otimes\cdots\otimes\mathcal{A}(A_1,A_2)\otimes\mathcal{A}(A_0,A_1)\to\mathcal{A}(A_0,A_n)$$

A more generic way to do this is indicated by a suggestive example of weakly enriching over a strict n-category, in which the action is given by an object that is shown to be a bimodule of the operad of associahedra and the associative operad. Recall that an operad is a monoid in the category of sequences in a symmetric (or n-fold monoidal) category with substitution product. Thus a (left) operad module is a sequence which the operad acts upon, i.e. $\boldsymbol{\mathcal{D}}$ is a left module of $\mathcal{C}$ if $\mathcal{C}(k)\times(\boldsymbol{\mathcal{D}}(j_1)\times\cdots\times\boldsymbol{\mathcal{D}}(j_k))\to\boldsymbol{\mathcal{D}}(n)$ where n is the sum of the j_i. The action obeys associativity and unit axioms.

Other questions are logically parallel to the proposed work, such as the open question of whether every loop space is modeled by an iterated monoidal category, or whether weak n-categories can precisely model homotopy n-type k-fold loop spaces for $k>n$.

For a full investigation of the most general case of enrichment over weak n-categories, it will be important to determine what sorts of operad-related structure might characterize (lax) enriched functors, natural transformations, and higher morphisms. This will be required to describe the overall categorical structure of the collection of enriched categories over weak n-categories. The clue the investigators plan to follow is in the structure of an operad based sequence of natural polytopes which are descendants of both the associahedra and the composihedra. These naturahedra parameterize higher enriched natural transformations. A construction exists, but the general structure is unknown. Convex hull realizations might allow computerized investigation of this structure.

2. Background

2.1. **Classical enrichment and n-categories.** The history of (weak) n-categories is quite tied up with enrichment. One way in which the concept of enrichment is central to category theory is that the category of categories, **Cat**, is actually enriched over itself. For every pair of categories there is a category of functors and natural transformations between them. In general a category enriched over **Cat** is called a 2-category and a strict n-category is defined recursively as a category enriched over the monoidal category of $(n-1)$-categories. By categorical dimension is meant the level of this enrichment. High dimension allows the enrichment process to be weakened. Weakening in this context refers to the transition from strictly associative composition of morphisms to the existence of higher morphisms that mediate associativity.

2.2. **Classical enrichment over braided categories.** Joyal and Street [27] studied enrichment over braided and symmetric categories and concluded that the category of enriched categories is merely monoidal when the base category is merely braided. If the base category is symmetric then the category of enriched categories is also symmetric. These results brought to mind the functor of delooping, for which infinite loop spaces or abelian topological groups play the role of symmetric monoidal categories. Indeed the classifying space of a symmetric category is known to be an infinite loop space. Stasheff [47] and MacLane [38] showed that monoidal categories are precisely analogous to 1–fold loop spaces. The first step in filling in the gap between 1 and infinity was made in [15] where it is shown that the group completion of the nerve of a braided monoidal category is a 2–fold loop space.

2.3. **Fiedorowicz's n-fold categories model loop spaces.** One major recent advance is the discovery of Balteanu, Fiedorowicz, Schwänzl and Vogt in [3] that the nerve functor on categories gives a direct connection between iterated monoidal categories and iterated loop spaces. In [3] the authors in their words, "pursue an analogy to the tautology that an n-fold loop space is a loop space in the category of $(n-1)$-fold loop spaces." The first thing they focus on is the fact that a braided category is a special case of a carefully defined 2-fold monoidal category. Based on their observation of the correspondence between loop spaces and monoidal categories, they iteratively define the notion of n-fold monoidal category as a monoid in the category of $(n-1)$-fold monoidal categories. In [3] a symmetric category is seen as a category that is n-fold monoidal for all n. The main result in that paper is that the group completion of the nerve of an n-fold monoidal category is an n-fold loop space. It is still open whether this is a complete characterization, that is, whether every n-fold loop space arises as the nerve of an n-fold monoidal category. Much progress towards the answer to this question was made by the original authors in their sequel paper, but the desired result was later shown to remain unproven. One of the future goals of the program begun here is to use weakenings or deformations of the examples of n-fold monoidal categories introduced here to model specific loop spaces in a direct way.

The connection between the n-fold monoidal categories of Fiedorowicz and the theory of higher categories is through the periodic table as laid out in [1]. Here Baez organizes the k-tuply monoidal n-categories, by which terminology he refers to $(n+k)$-categories that are trivial below dimension k. The triviality of lower cells allows the higher ones to compose freely, and thus these special cases of $(n+k)$-categories are viewed as n-categories with k multiplications. Of course a k-tuply monoidal n-category is a special k-fold monoidal n-category. The specialization results from the definition(s) of n-category, all of which seem to include the axiom that the interchange transformation between two ways of composing four higher morphisms along two different lower dimensions is required to be an isomorphism. The property of having iterated loop space nerves held by the k-fold monoidal categories relies on interchange transformations that are not isomorphisms. If those transformations are indeed isomorphisms then the k-fold monoidal 1-categories do reduce to the braided and symmetric 1-categories of the periodic table. Whether this continues for higher dimensions, yielding for example the sylleptic monoidal 2-categories of the periodic table as 3-fold monoidal 2-categories with interchange isomorphisms, is yet to be determined.

2.4. **Enrichment over n-fold monoidal categories yields n-1-fold monoidal categories.** The results about iterated enrichment over iterated monoidal categories reviewed here have already been reported in [16] and [17]. For enrichment to accurately represent the formation of the topological classifying space, then at each stage of successive enrichments, the number of monoidal products should decrease and the categorical dimension should increase, both by one. The immediate question is whether the delooping phenomenon happens in general for k–fold monoidal categories. The answer is yes, once enriching over a k–fold monoidal category is carefully defined in [16]. The definition also provides for iterated delooping, and all the information included in the axioms for the k–fold category is exhausted in the process.

The concept of higher dimensional enrichment is important in its relationship to double, triple and further iterations of delooping. In [17] we define $\mathcal{V}$-$(n+1)$-categories as categories enriched over $\mathcal{V}$-n-Cat, the $(k-n)$-fold monoidal strict $(n+1)$-category of $\mathcal{V}$-n-categories where $k > n \in \mathbf{N}$. $\mathcal{V}$-1-Cat is just the usual $\mathcal{V}$-Cat.

The concept of a k-fold monoidal strict n-category is easy enough to define as a tensor object in a category of $(k-1)$-fold monoidal n-categories with cartesian product. Thus the products and accompanying associator and interchange transformations are strict n-functors and n-natural transformations respectively. That this sort of structure ($(k-n)$-fold monoidal strict $n+1$ category) is possessed by $\mathcal{V}$-n-Cat for $\mathcal{V}$ k-fold monoidal is shown in [17]by presenting a full inductive proof covering all n, k. In general the decrease is engineered by a shift in index–we define new products $\mathcal{V}$-n-Cat$\times\mathcal{V}$-n-Cat $\to$ $\mathcal{V}$-n-Cat by using cartesian products of object sets and letting hom-objects of the ith product of enriched n-categories be the $(i+1)$th product of hom-objects of the component categories. Symbolically,

$$(\mathcal{A} \otimes_i^{(n)} \mathcal{B})((A,B),(A',B')) = \mathcal{A}(A,A') \otimes_{i+1}^{(n-1)} \mathcal{B}(B,B').$$

The superscript (n) is not necessary since the product is defined by context, but we often insert it to make clear at what level of enrichment the product is occurring. We complete the process by defining the necessary natural transformations for this new product as "based upon" the old ones.

2.5. **Delooping example: group torsors and tensored enriched categories.** Group torsors are examples of enriched categories over a monoidal category, in this case the group itself. It has recently been noticed that the category of torsors possesses a classifying space which is precisely the classifying space of the group. This suggests that the functor in question should actually be the tensored enrichment functor. Another direction to go in is that of $\mathcal{V}$–Mod, the category of $\mathcal{V}$–categories with $\mathcal{V}$–modules as morphisms. $\mathcal{V}$–Mod should also be investigated for the case of $\mathcal{V}$ k–fold monoidal. The same is true of $\mathcal{V}$–Act, the category of categories with a $\mathcal{V}$ action. These both overlap enriched tensored categories in important ways.

Given a group G:

(1) Let $\underline{G}$ be the category whose objects are elements of G and whose only morphisms are identity arrows. $\underline{G}$ is monoidal with $\otimes$ given by the group operation and $I = e$.
(2) Let Tor(G) be the category of G-torsors and G-equivariant maps (respect action.) A G-torsor $\mathcal{B}$ is a set $\mathcal{B}$ with effective G-action; that is for all $x, y \in \mathcal{B}$ there exists a unique $g_{xy} \in G$ such that $g_{xy}x = y$. First we notice that $\mathcal{B}$ is a $\underline{G}$-category. $\mathcal{B}(x,y) = g_{xy}$ and equivariant maps are enriched functors. Secondly we notice that G itself is a G-torsor; i.e. $\underline{G}$ is closed. Enriched functors $G \to G$ are the elements of G. We denote by $\overline{G}$ this single category subcategory of Tor(G). We also note that a $\underline{G}$-category tensored over $\underline{G}$ is precisely a G-torsor, since letting $x \otimes g = gx$ gives $\mathcal{B}(x \otimes g, z) = \underline{G}(g, \mathcal{B}(x,z))$.
(3) We use the fact every G-torsor is isomorphic to G. Thus $\overline{G}$ is a skeleton of Tor(G). Recall that $Nerve\,\mathrm{Tor}(G) = Nerve\overline{G} = BG = B(Nerve\underline{G})$ and thus we have that $B(Nerve\underline{G}) \subseteq Nerve(\underline{G}$-Cat). The subset relation becomes an equality when we restrict to $\underline{G}$-categories tensored over $\underline{G}$.

Further study of the general case should attempt to elucidate the relationship of the nerves of the n-categories in question. For instance, we would like to know the relationship between $\Omega Nerve(\mathcal{V}\text{-Cat})$ and $Nerve(\mathcal{V})$. This would even be quite interesting in the case of symmetric $\mathcal{V}$ where the nerve is an infinite loop space. It would be nice to know if there are symmetric monoidal categories whose nerves exhibit periodicity under the vertically iterated enrichment functor.

2.6. **Batanin's weak n-categories model homotopy n-types.** A further refinement of higher categories is to require all morphisms to have inverses. These special cases are referred to as n-groupoids, and since their nerves are simpler to describe it has long been suggested that they model homotopy n-types through a construction of a fundamental n-groupoid. This has in fact been shown to hold in Tamsamani's definition of weak n-category [51], and in a recent paper by Cisinski to hold in the definition of Batanin as found in [5]. A homotopy n-type is a topological space X for which $\pi_k(X)$ is trivial for all $k > n$. Thus the homotopy n-types are classified by π_k for $k \le n$.

It has been suggested that a key requirement for any useful definition of n-category is that a k-tuply monoidal n-groupoid be associated functorially (by a nerve) to a topological space which is a homotopy n-type and a k-fold loop space [1]. The loop degree will be precise for $k < n+1$, but for $k > n$ the associated homotopy n-type will be an infinite loop space. This last statement is a consequence of the stabilization hypothesis , which states that there should be a left adjoint to forgetting monoidal structure that is an equivalence of $(n+k+2)$-categories between k-tuply monoidal n-categories and $(k+1)$-tuply monoidal n-categories for $k > n+1$. This hypothesis has been shown by Simpson to hold in the case of Tamsamani's definition [45]. For the case of $n = 1$ if the interchange transformations are isomorphic then a k-fold monoidal 1-category is equivalent to a symmetric category for $k > 2$. With these facts in mind it is possible that if we wish to precisely model homotopy n-type k-fold loop spaces for $k > n$ then we may need to consider k-fold as well as k-tuply monoidal n-categories.

3. Strict enrichment in one dimension and Young diagrams

The categories here are the iterated monoidal categories of Balteanu et.al. [3]. Recall that the expanded definition of these posits the existence of multiple products in the category. There are n distinct multiplications

$$\otimes_1, \otimes_2, \ldots, \otimes_n : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$$

for each of which the associativity pentagon commutes. $\mathcal{V}$ has an object I which is a strict unit for all the multiplications. For each pair (i, j) such that $1 \le i < j \le n$ there is a natural transformation

$$\eta^{ij}_{ABCD} : (A \otimes_j B) \otimes_i (C \otimes_j D) \to (A \otimes_i C) \otimes_j (B \otimes_i D).$$

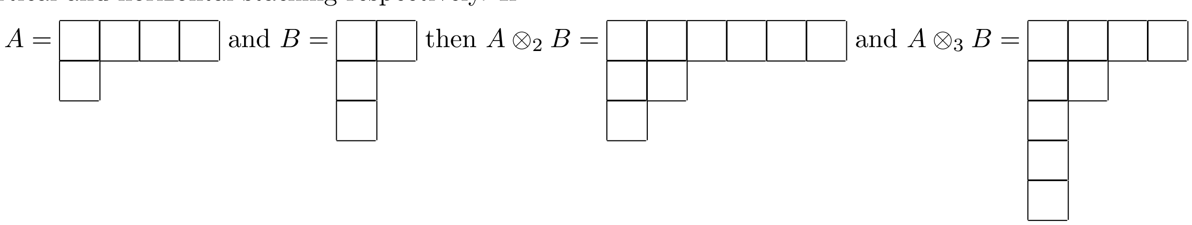
These natural transformations η^{ij} are subject to the unit and associativity conditions.

Braided categories arise as special 2-fold monoidal categories and symmetric categories as n-fold monoidal for all n. The general results in regard to enrichment are as described above: enrichment decreases the monoidalness. The question is whether the enrichment always models the delooping. In other words we have the following diagram of functors:

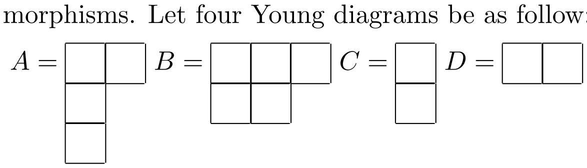
$$\begin{array}{ccc} n\text{-fold monoidal categories } \mathcal{V} & \xrightarrow[\textit{(Geom. real. of Nerve)}]{\textit{Class. space}} & n\text{-fold loop spaces } \Omega^n X \\ {\scriptstyle \textit{Enrich}} \Big\uparrow {\scriptstyle \mathcal{V}\to\mathcal{V}-\text{Cat}} & & \Big\downarrow {\scriptstyle \Omega} \\ (n+1)\text{-fold monoidal categories } \mathcal{V} & \xrightarrow[\textit{Class. space}]{} & \Omega^{n+1} X \end{array}$$

The large questions inspired by this are 1) in what sense does the square commute, and 2) can we find suitably adjoint functors to the shown arrows, and in which cases are there underlying bijections. The open question in the latter case is in regard to whether every iterated loop space can be modeled precisely by an appropriate category. The answers to both questions together promise to make much more clear how well enrichment works as categorical delooping.

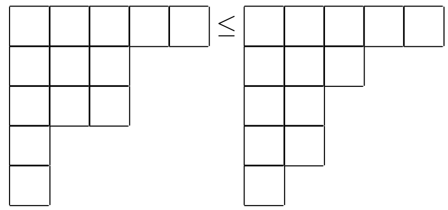
3.1. **Enrichment over example iterated monoidal categories.** In [19] example categories have been found with objects the n-dimensional Young diagrams. In the 2-dimensional case we let $\otimes_3$ be the product which adds the heights of columns of two diagrams, $\otimes_2$ adds the length of rows. We often refer to these as vertical and horizontal stacking respectively. If



We can take as morphisms the totally ordered structure of the Young diagrams given by lexicographic ordering. Thus we may retain the lexicographic max as $\otimes_1$, and will refer to the entire category simply as the category of Young diagrams. The unit object is the zero diagram. The iterated monoidal structure is the existence of interchange morphisms. Let four Young diagrams be as follow:



Then the fact that $(A \otimes_3 B) \otimes_2 (C \otimes_3 D) \le (A \otimes_2 C) \otimes_3 (B \otimes_2 D)$ appears as follows:



Higher dimensions are also described in [19]. It is shown that the category of n-dimensional Young diagrams with stacking products in each dimension constitutes an n-fold monoidal category. Two questions are immediate: what is a simple description of $\mathcal{V}$-Cat for these examples, and which of these examples can precisely model loop spaces. The latter question brings us back to the open question of loop space modeling.

3.2. **Decorated Young diagram categories as loop space models.** Let the non-commutative category of Young diagrams over $S = \Omega^2 X$ a double loop space be the diagrams with blocks labeled by elements of S :

a	b	1	f	g	h
c	1				
d					

Take as the two products of a 2-fold monoidal category the vertical and horizontal stacking as shown above and let the ordering morphisms of the usual category of Young diagrams be altered as follows:

(1) Retain only morphisms between A and B such that A represents a map $S^2 \to X$ homotopic to the map represented by B. (Representation here is the obvious compositions of the maps in each block.)
(2) Further retain only morphisms between A and B for which there is an injective map
$$f : \{x \in S| \; \boxed{x} \in A\} \to \{y \in S| \; \boxed{y} \in B\}.$$
such that x is homotopic to $f(x)$. Now expand the existing morphisms by labelling each with a function f which obeys these requirements.
(3) For the special cases of A, B both a single row or A, B both a single column: retain only the morphisms for which the labelling function f respects the linear ordering of the loop space elements decorating A. Thus there will be no morphism from $\boxed{a}\boxed{b}$ to $\boxed{b}\boxed{a}$.

The conjecture for this category is that the geometric realization of the nerve is homotopy equivalent to the original double loop space. For two points a and b in the double loop space S, ab is homotopic to ba; there is a path in S between ab and ba. This is preserved by the just described construction. The path in S between them is found in the nerve of the non-commutative category of Young diagrams over S since there is a map from $\boxed{a}\boxed{b}$ to $\begin{array}{|c|}\hline a\\\hline b\\\hline\end{array}$ as well as a map from $\boxed{b}\boxed{a}$ to $\begin{array}{|c|}\hline a\\\hline b\\\hline\end{array}$. The usual Eckmann-Hilton proof that vertical and horizontal composition in a double loop space turn out to be equivalent under homotopy (and abelian) is subverted since although there is a map from $\begin{array}{|c|}\hline a\\\hline b\\\hline\end{array}$ to $\begin{array}{|c|c|}\hline a & 1\\\hline 1 & b\\\hline\end{array}$ there is not an inverse map since the lexicographic ordering makes the 4-block diagram larger. Here the block labeled by "1" (= the constant map) is not the unit in our category -recall that that the zero diagram is our unit.

The next question we plan to attack then will be what can be said of the classifying space (nerve) of the category of enriched (tensored) categories over such a loop space model.

4. Weak enrichment over n-fold monoidal categories in one dimension

There may be some considerable value in investigating the one-dimensional analogs of the full n-categorical comparison of delooping and enrichment. One dimensional weakened versions of enriched categories have been well-studied in the field of differential graded algebras and A_∞-categories, the many object generalizations of Stasheff's A_∞-algebras [47].

An A_∞-category category is basically a category "weakly" enriched over chain complexes of modules, where the weakening in this case is accomplished by summing the composition chain maps to zero (rather than by requiring commuting diagrams). It is also easily described as an algebra over a certain operad. We plan to generalize this idea to any base category with sufficient structure.

Recently operads in n-fold monoidal categories have been defined [19]. There are many open problems suitable for student research regarding the combinatorics and computation of operads and their modules in the categories of Young diagrams. Examples and early results are in [19]. Weak enrichment can be described in terms of the actions of operad algebras and modules (operads and operad algebras are special operad modules). The composition maps parameterized by an operad module $\boldsymbol{\mathcal{C}}(n)$ appear as follow:

$$\boldsymbol{\mathcal{C}}(n) \otimes \mathcal{A}(A_{n-1}, A_n) \otimes \cdots \otimes \mathcal{A}(A_1, A_2) \otimes \mathcal{A}(A_0, A_1) \to \mathcal{A}(A_0, A_n)$$

We would like of course to understand connections between weak enrichment over decorated Young diagrams and delooping. Also of great interest would be a theory of tensored weakly enriched categories over n-fold monoidal categories.

5. Weak enrichment over strict n-categories

5.1. **Composihedra and operad bimodules: characterizing weak enrichment.** This new family of polytopes is defined based upon the definitions of the associahedra (Stasheff, [47]) and the multiplihedra (Iwase and Mimura , [26], Boardman and Vogt, [9]). These are shown to comprise a left module over the associahedra, and a right module over the associative operad. The elements of the composihedra are shown to underly the commutative pasting diagrams in the naive definition of a category weakly enriched over a monoidal strict n-category.

Recall that the multiplihedra are a bimodule of polytopes over the associahedra. They are also the underlying complexes of the operad of spaces of $\{X, Y\}$ bi-colored trees with n leaves and internal edges of length in $[0, 1]$ defined by Boardman and Vogt. They characterize A_n maps between two A_n spaces $X \to Y$. The composihedra result from identifications made in the multiplihedra, where each copy of the n^{th} associahedron that is found entirely in the domain (colored by X) is collapsed to a point. This is equivalent to considering the special case in which X is associative. It also lets us write a recursive definition of the composihedra $\mathcal{C}K(n)$ We use (order preserving) partitions of strings of elements of the free monoid on an alphabet. Equivalent strings are those that give the same word when concatenated. For example $a, bc, de \sim a, \_, bcde$ since both give $abcde$ upon concatenation.

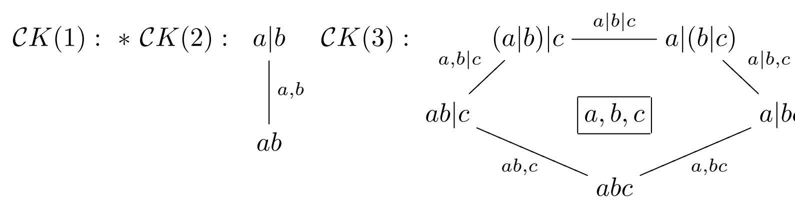
Definition: Let $\mathcal{C}K(1) = *$. Given a string of $n > 1$ (comma delimited) letters as a label, the n^{th} composihedra $\mathcal{C}K(n)$ is the cone on $L(n)$ where $L(n)$ is cell complex that is the union of $n + 2^{n-1} - 2$ facets. These include the "upper" facets: $(n-1)$ copies of $\mathcal{C}K(n-1)$ labeled by a choice of a concatenated consecutive pair in the string of letters, such as a, bc, d, e labeling a copy of $\mathcal{C}K(4)$ in $\mathcal{C}K(5)$. Most importantly is the existence of $2^{n-1} - 1$ "lower" facet inclusions: $\mathcal{K}(k) \times (\mathcal{C}K(j_1) \times \cdots \times \mathcal{C}K(j_k)) \to \mathcal{C}K(n)$ where n is the sum of the j_i, each $j_i \geq 1$, and where $2 \leq k \leq n$. This is the left module structure. These lower facets can be labeled by partitions of the string of n letters, excluding the trivial partition, where k is the number of divisions in the partition and j_i is the number of letters in the i^{th} division. E.g., in $\mathcal{C}K(5)$, the label $a, b|c, d, e$ would be applied to the facet $\mathcal{K}(2) \times (\mathcal{C}K(2) \times \mathcal{C}K(3))$. Faces of the facets will be labeled by either concatenating a pair of consecutive comma delimited letters (or recursively, words), by inserting a pair of parentheses around two or more of the divisions, or by sub-partioning a division. In the latter case a pair of parentheses must also be inserted around what was the old division, unless it was the only (trivial) one. Faces of facets with the same label are identified to produce $L(n)$.

Given a string of $n > 1$ letters, vertices (0-cells) correspond to completely bracketed, completely partitioned, equivalent strings of $k \leq n$ words. Edges (1-cells) correspond to either an incomplete bracketing of a string that has as its completions the two vertices it connects, or a concatenation of two words separated only by a vertical bar. The number of vertices in the n^{th} composihedron is the sequence that begins:

$$1,\ 2,\ 5,\ 15,\ 51,\ 188,\ 731,\ 2950,\ \ldots.$$

This sequence is the binomial transform of the Catalan numbers. It can be described in several ways. The closest description, which gives an alternate way of indexing the vertices, is that the sequence gives the number of binary trees of weight n where leaves have positive integer weights. This is the non-commutative non-associative version of partitions of n.

Example: Here are the first few composihedra with vertices labeled by bracketed word strings. Notice how the n^{th} composihedron is based on the n^{th} associahedron. The Schlegel diagram is shown for $\mathcal{C}\mathcal{K}(4)$, in which the facet labels are boxed and only some of the edges are labeled.



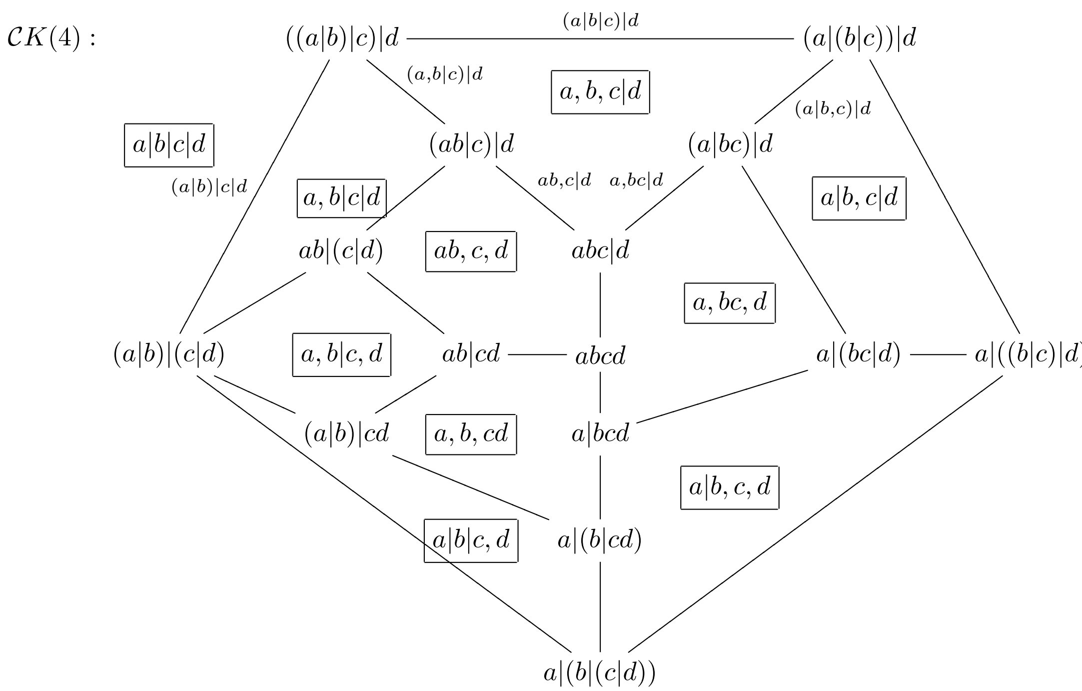


Figure 1 shows the last of the above examples, $\mathcal{C}\mathcal{K}(4)$, drawn 3-dimensionally. The bold outlined face pentagon in the upper right is the copy of $\mathcal{K}(4)$ which appears on the outside of the above diagram.

The definition of a category weakly enriched over a monoidal strict n-category $\mathcal{V}$ leads to a study of weak n-category theory restricted to horizontal compositions.

In an enriched category $\mathcal{A}$ (over $\mathcal{V}$) the role of composition is taken over by special morphisms in the monoidal category $\mathcal{V}$. A string of these hom-objects (such as the string of length two in the domain of the composition morphism $M : \mathcal{A}(B,C) \otimes \mathcal{A}(A,B) \to \mathcal{A}(A,C)$) will be called composable if they can be reduced to a single hom-object by repeated uses of M. Of course the parenthization of the original string matters. Keep in mind then also the associator α, used to get from one parenthization to another. For a string of length n one can draw the associahedron $\mathcal{K}(n)$ and put the various parenthizations at the vertexes, and the associators on the edges.

When enriching over a monoidal strict n-category $\mathcal{V}$, composition morphisms $\boldsymbol{\mathcal{M}} : \boldsymbol{\mathcal{U}}(B,C) \otimes \boldsymbol{\mathcal{U}}(A,B) \to \boldsymbol{\mathcal{U}}(A,C)$ are 1-cells. The pentagon they are usually required to satisfy exactly for each triple of objects can instead be filled in with an (invertible) 2-cell $\boldsymbol{\mathcal{M}}_2$. To save space "$\bullet\ \bullet \to \bullet$" will represent $\boldsymbol{\mathcal{M}} : \boldsymbol{\mathcal{U}}(B,C) \otimes \boldsymbol{\mathcal{U}}(A,B) \to \boldsymbol{\mathcal{U}}(A,C)$. Thus the first location in the pentagon stands in for

$$(\boldsymbol{\mathcal{U}}(C,D) \otimes^{(n-1)} \boldsymbol{\mathcal{U}}(B,C)) \otimes^{(n-1)} \boldsymbol{\mathcal{U}}(A,B)$$

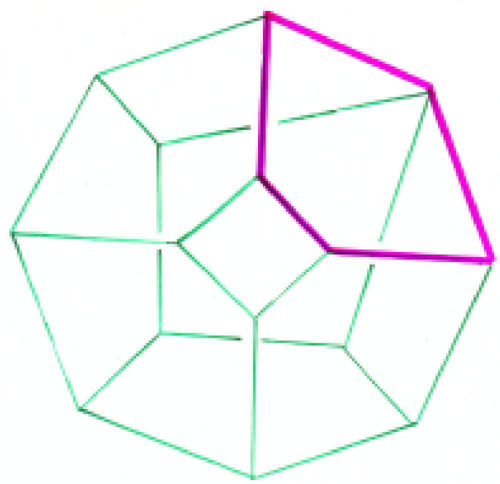
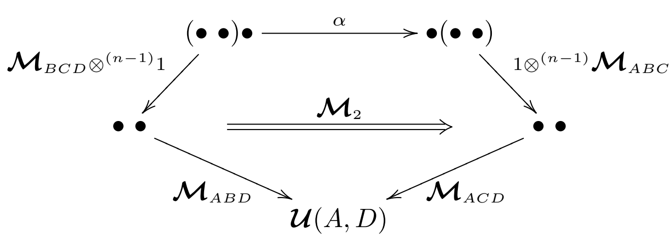
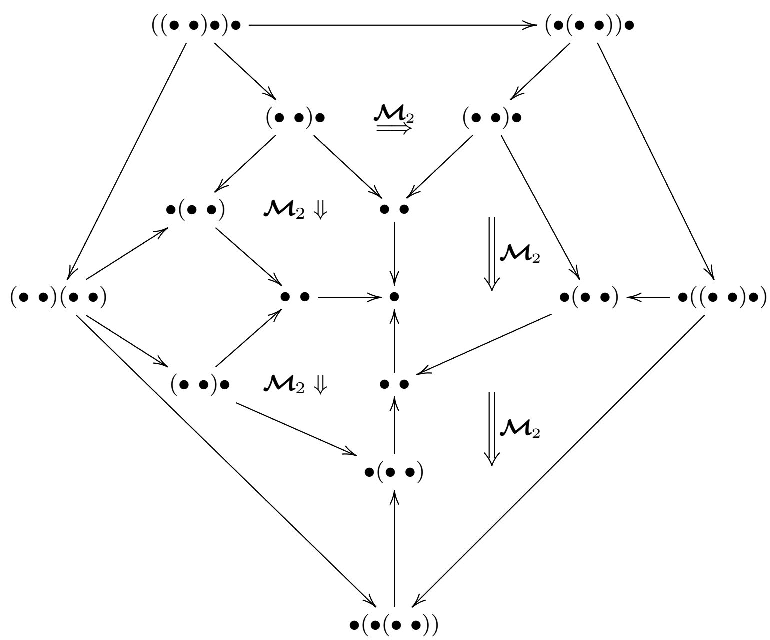


Figure 1. $\mathcal{C}K(4)$

and the others can be easily determined as range and domain of the arrows.



For each quadruple of objects $\mathcal{M}_2$ is required to participate in a higher law. Draw the pentagonal diagram of $\mathcal{K}(4)$ with vertices labeled by complete bracketings of a string of composable hom-objects, and compose each vertice by use of $\mathcal{M}$ to a common final hom-category. What this does is to subdivide $\mathcal{K}(4)$ into 2-cells which are again filled with instances of $\mathcal{M}_2$.



There are exactly two 2-dimensional paths that make up the front and back of the polyhedron. The division between the two is along the line of the quickest path to the center from the upper left, as well as the path from the upper left to the center which goes through the lowest location. Between the two there should now exist an enriched modification $\boldsymbol{\mathcal{M}}_3$.

The series of k-cells $\boldsymbol{\mathcal{M}}_k$ fill in polytopal diagrams that are in fact the composihedra: the boundary of $\mathcal{C}K(i) = s(\boldsymbol{\mathcal{M}}_{i-1}) \sqcup t(\boldsymbol{\mathcal{M}}_{i-1})$ where s and t denote the source and target of the morphism in question.

In general for each n-tuple of objects there exists an invertible $(n-2)$ -cell $\boldsymbol{\mathcal{M}}_{(n-2)}$, until at last for each set of $n+3$ objects $\boldsymbol{\mathcal{M}}_{n+1}$ is an identity morphism – i.e. the last diagram involving several instances of $\boldsymbol{\mathcal{M}}_n$, in the form of $\mathcal{C}K(n+2)$, is required to commute.

The quick way to describe this series of commuting diagrams is in terms of an operad module action. We can define an action of the operad module of spaces $\mathcal{C}K(n)$ on $\mathcal{V}$ (by describing the category of colored trees for which the $\mathcal{C}K(n)$ are the classifying spaces) so that the higher compositions are given by

$$\boldsymbol{\mathcal{M}}_{(n+1)} : \mathcal{C}K(n) \times \mathcal{A}(A_{n-1}, A_n) \otimes \cdots \otimes \mathcal{A}(A_1, A_2) \otimes \mathcal{A}(A_0, A_1) \to \mathcal{A}(A_0, A_n).$$

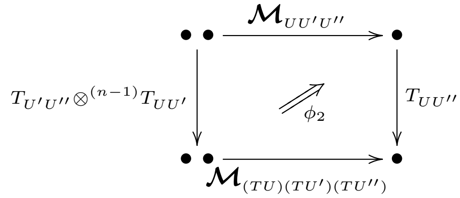
5.2. **A_n maps into a loop space and enriched categories over a strict fundamental n-category.** Recall that A_n spaces are not quite topological groups, or even monoids. Their multiplication is only associative up to homotopy. Stasheff showed that we can recognize them by finding an action of the associahedra, $\mathcal{K}(i)$ for $i = 1$ to n. The classic example of course is the loop space of a space, which is an A_n space for all n. Recall that A_n maps are not quite homomorphisms. They respect the multiplication of the A_n spaces up to homotopy. Iwase and Mimura [26] show that these maps are characterized by actions of the multiplihedra. Boardman and Vogt [9] described the spaces of trees corresponding to the associahedra and the multiplihedra. When we assume that the range is an associative H-space, the multiplihedra collapse to become the associahedra as shown by Stasheff in [48]. When both the range and domain are associative the multiplihedra become the cubes as shown in [9]. When the domain alone is an associative H-space, however, the multiplihedra become the composihedra.

The strict fundamental n-category of a topological space has objects points of the space, 1-cells paths up to reparametrization, and higher cells homotopies up to reparametrization.

In the case of A_n maps from topological groups, it is conjectured that maps into a loop space which preserve the group structure only up to homotopy are in bijection with weakly enriched categories over the fundamental n-category in question. Given $f : X \to Y$ an A_n map with X a topological group, we can define $\mathcal{A}$ weakly enriched over the strict fundamental n-category of Y by taking the objects of $\mathcal{A}$ to be the points of X, $\mathcal{A}(a, b) = f(b^{-1}a)$ and the composition $\boldsymbol{\mathcal{M}}_1$ to be the homotopy from $f(c^{-1}b)f(b^{-1}a)$ to $f(c^{-1}a)$. The higher compositions are the higher homotopies, and the axioms are clearly obeyed.

5.3. **Naturahedra and enriched n-natural transformations.** We have described weak enrichment over a monoidal strict n-category as base. What we are really interested in is the structure of the category of all weakly enriched categories given the strcture of the base. To study this we need to understand the maps of weakly enriched categories in order to talk about the categorical structure of their compositions as well as extra monoidal structure inherited from the base. The usual concept of enriched functor [30] is easily weakened here. Lax enriched n-functors are defined as follows:

For a strict n-category $\mathcal{V}$ and two weak $\mathcal{V}$-categories $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{W}}$ a lax $\mathcal{V}$-*functor* $T : \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{W}}$ is a function on objects $|\boldsymbol{\mathcal{U}}| \to |\boldsymbol{\mathcal{W}}|$ and a family of 1-cells in $\mathcal{V}$; $T_{UU'} : \boldsymbol{\mathcal{U}}(U, U') \to \boldsymbol{\mathcal{W}}(TU, TU')$. Then for each set of $k > 2$ objects there is a k-cell ϕ_k that fills in a polytope diagram made by taking a right prism of the polytope $\mathcal{C}K(k)$. Since the $T_{UU'}$ are enriched functors the square they are usually required to satisfy exactly can instead be filled in with an (invertible) enriched n-natural transformation ϕ_2. This square is the prism on $\mathcal{C}K(2)$.



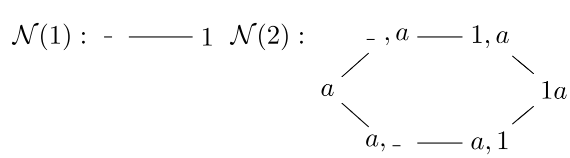
The 2-cell ϕ_2 is required to participate in the following diagram: draw the pentagonal prism of $\mathcal{C}K(3)$, faces filled in with $\boldsymbol{\mathcal{M}}_2$ on both pentagonal ends and with ϕ_2 on the sides. Then the prism itself is filled in with the enriched modification ϕ_3. This process continues, and ϕ_{n+1} is the identity.

A family of polytopes is developed which underlie the naive axioms for higher transformations between categories weakly enriched over a strict monoidal n-category. First we discuss the polytopes, since the important question is just what sort of operadic structure they define. Then in the next subsection we justify the discussion by demonstrating how they characterize lax enriched natural transformations, modifications, and higher transformations.

The strings of words in the composihedra are assumed to be made up of nonempty words, and the letters are assumed to be non-central. This suggests two ways to generalize the indexing of the vertices of the composihedra. One is to consider bracketed strings that include a certain number of instance of the empty word. The simplest next step is to allow at most one instance of the empty word. This corresponds to weighted trees of a given weight that may include at most one leaf of weight zero. The number of these for weight n form a new sequence that begins $1,\ 3,\ 10,\ 39,\ 165,\ \ldots$ The general formula for this sequence is unknown but would make a good problem for an undergraduate project. The other generalization is to consider including in the original string of letters a central generator of the monoid. It is denoted by 1. Now there are words that are considered equal, such as $ab1c = abc1$ and for the sake of simplicity in the definition these equal words are identified and written with the unit first, $1abc$. The number of bracketed strings of words based on n letters including one central letter is a sequence that begins $1,\ 3,\ 11,\ 45,\ 195,\ 873,\ \ldots$ This is the same beginning as the sequence formed by taking the binomial transform of the central binomial coefficients. To prove the conjecture that the two sequences are identical is another good undergraduate problem. Neither generalization of combinatorial indexing leads immediately to a sequence of polytopes. However, both indexes contain the examples in which the new element, respectively the empty word and the central generator, have yet to be concatenated with any neighboring words in the sequence. The number of these examples is the same for both cases, and thus by connecting the corresponding instances of these cases a polytope is formed.

Given a string of $n-1$ letters the n^{th} naturahedron is a cell complex whose vertices are indexed by completely bracketed distinct equivalent strings of $k \le n$ words based on the original string of $n-1$ letters together with either a single copy of the empty word or a single central element 1. In addition to edges inherited from included composihedra are those where the empty word and the unit 1 occupy the same location in the bracketed strings which are the vertices connected by that edge.

Example: Here are the first few naturahedra. Notice how the n^{th} naturahedron is based on the $(n-1)^{st}$ composihedron.



$\mathcal{N}(3)$:

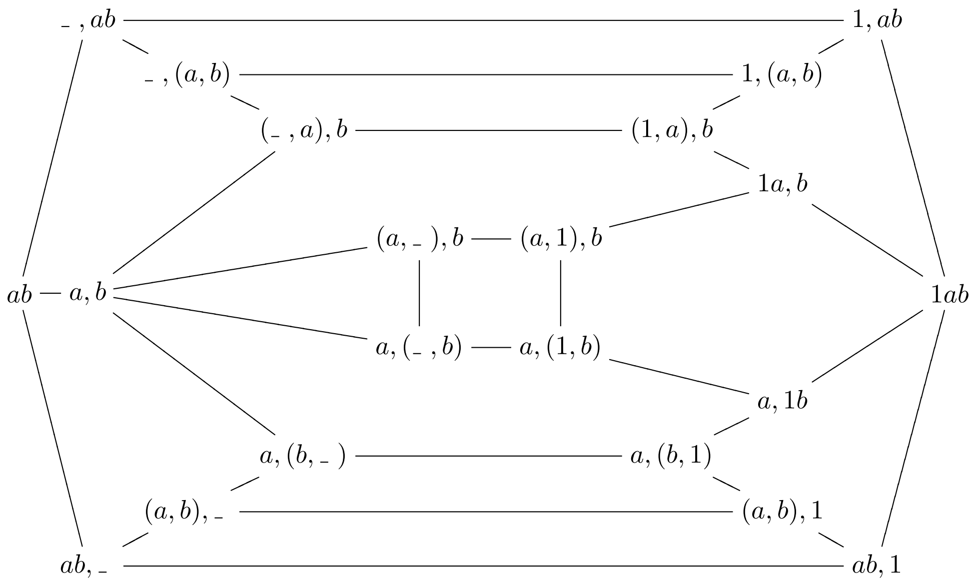


Figure 2 shows the last of the above examples, $\mathcal{N}(3)$, drawn 3-dimensionally. The bold edge in the upper center is the copy of $\mathcal{C}K(2)$ which appears at the far left of the above diagram.

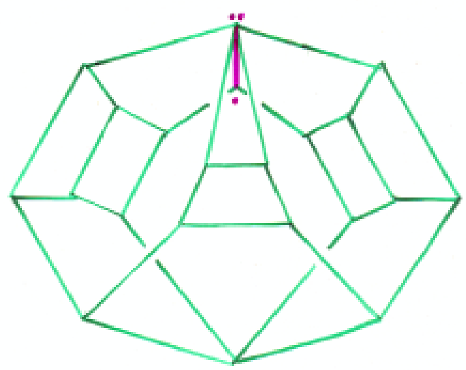


FIGURE 2. $\mathcal{N}(3)$

Theorem: (draft) The n^{th} naturahedron has boundary topologically equivalent to the $(n-1)$-sphere. In fact it is the boundary of a convex n dimensional polytope.

Proof sketch: The case of the polyhedron $\mathcal{N}(3)$ follows from the Steinitz theorem which states that any simple planar 3-connected graph can be realized as a convex polyhedron. The general case will follow from a complete recursive combinatorial description, and use of the fact that the naturahedra are basically the joining of two generalized copies of composihedra.

Theorem: (draft) The naturahedra posess a parity complex structure.

Proof sketch: This will follow from the previous parity structures shown for the associahedra and composihedra. Also important may be the well known theorem that any polytopal directed graph has exactly one sink and one source [25].

5.4. **Using Naturahedra to describe lax higher enriched transformations.** All strict higher morphisms of enriched n-categories have a shared form of their axiomatic commuting diagram, as seen in [17]. Thus only a single new sequence of morphisms is required to describe lax enriched k-cells between strict enriched $(k-1)$ cells between lax enriched n-functors. For this reason the naturahedra are described as having the "universal" property of playing the same role at each categorical dimension. Here is a draft of the theorem to be proven.

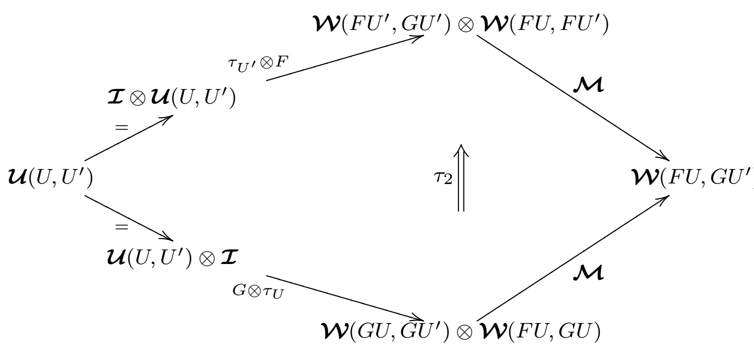
Theorem: (draft) The sources and targets of the mediating morphisms of a higher lax enriched transformation are together given by the boundary of the appropriate naturahedron.

The proof will use a translation from the combinatorial description to a categorical one.

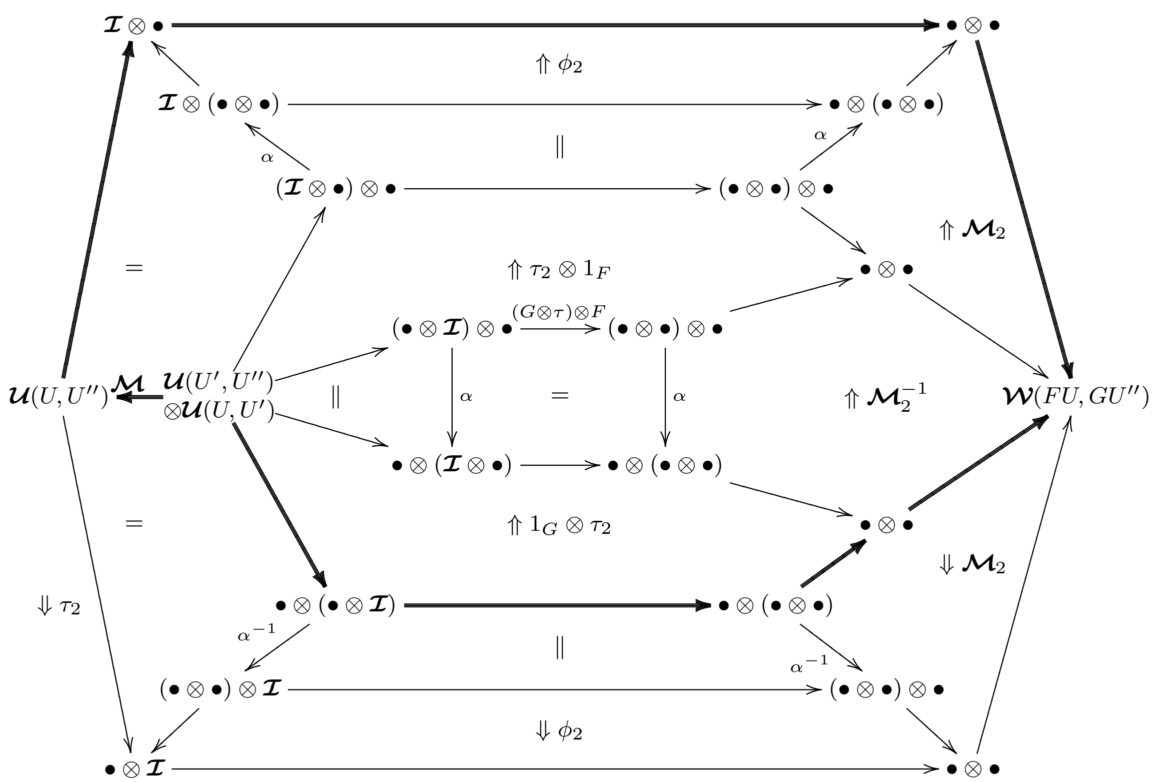
A weak $\mathcal{V}$-*n:k-cell* τ between $(k-1)$-cells ψ^{k-1} and ϕ^{k-1} is a function sending each $U \in |\boldsymbol{\mathcal{U}}|$ to a 1-cell in $\mathcal{V}$ with domain the unit object $\boldsymbol{\mathcal{I}}$. For instance a weak $\mathcal{V}$-natural transformation $\tau : F \to G$ is a family of 1-cells:

$$\tau_U : \boldsymbol{\mathcal{I}} \to \boldsymbol{\mathcal{W}}(FU, GU).$$

Rather than a commuting diagram there is a sequence of higher enriched cells that mediate the commutativity. The source and target of these form the boundaries of naturahedra. The enriched k-cell that fills the polytope diagram described by $\mathcal{N}(k)$ will be called τ_k. This process is illustrated here for the first few steps. Here is shown the domain and range for τ_2.



Now τ_2 is required to obey a commuting diagram of its own, or only to obey it up to a further higher morphism called τ_3. For brevity the superscripts showing dimension are omitted. The outermost hexagon is precisely the above diagram for τ_2. The bold arrows demarcate the division of source and target.



Once again this process will continue until at the top dimension $s(\tau_n) \sqcup t(\tau_n)$ is the boundary of $\mathcal{N}(n)$. Then τ_n will be required to obey a commuting diagram which has the form of $\mathcal{N}(n+1)$.

There are several secondary questions that immediately arise. The first regards the structure of the lax enriched morphisms just defined. It is likely that one of two things is true: either they form a strict n-category or a weakly enriched n-category. This needs to be investigated, especially since there is an unlikely third possibility that they conform to the axioms of one of the existing definitions of weak n-category. Another question concerns the correct way in which to strengthen lax transformations into equivalencies, in order to talk about equivalent weak enriched categories. Finally, of course, is the question of a theory of weak enriched limits. This will draw from the established theories of enriched (filtered) limits and lax limits (bilimits.)

5.5. **Enriching over strict k-fold monoidal n-categories.** So far we have discussed and given an example of weak enrichment over a monoidal strict n-category as base. If the base category is symmetric then we might expect the category of enriched categories to display a symmetry as well. If the base is k-fold monoidal then the category of enriched categories might be expected to be $k-1$-fold monoidal. These conjectures are based on the case of strict enrichment studied in detail in [17]. We also need to determine exactly what is a tensored category (weakly) enriched over a strict n-category? The first step would be to work out the definition for strict enrichment, which should not be difficult.

5.6. **Convex hull realizations.** As a tool for experimenting with multiplihedra and composihedra we generalize the construction of Loday [36] of convex hulls for the associahedra. It remains to be proven that these generalizations always hold. As a further tool we hope to also find realizations of the naturahedra.

The construction of Loday is as follows: given an n-leaved binary tree t we get a point $M(t) \in \mathbb{R}^{n-1}$ by calculating a coordinate from each trivalent vertex from left to right by multiplying the number of leaves in the subtree supported by the left branch times the number of leaves in the subtree to the right. The convex hull of the points $\{M(t)|t \in T(n)\}$, where $T(n)$ is the set of n-leaved binary trees, is the n^{th} associahedron.

Now given a X,Y-colored n-leaved tree t and a constant $q \in (0,1)$ we get $M'(t) \in \mathbb{R}^{n-1}$ by following the same procedure as above to find the coordinate for each trivalent vertex, but if the vertex is colored by the domain X, then the coordinate is multiplied by q. The convex hull of all the $M'(t)$ for n-leaved bi-colored trees is demonstrated to be the n^{th} multiplihedron. The full proof may use the description of the multiplihedra as given in [43]. The collapses of the multiplihedron are accomplished by considering the limits as $q \to 1$ and as $q \to 0$. The former gives the associahedron, since it restores Loday's construction, but the latter gives the composihedron since it does not differentiate between changes in the tree structure of the portion colored by X. The main value of this description of the convex hulls is that it can hopefully be generalized to find convex hull descriptions of the naturahedra. These will be quite useful, as were the composihedra descriptions, in guessing the operad structure of the naturahedra as a whole. Figure 3 shows the computer generated Schlegel diagram of $\mathcal{C}K(5)$ using the described q-algorithm and the program polymake.

6. Weak enrichment over weak n-categories

6.1. **Weak n-categories as operad algebras, weak enrichment using operad bimodules.** Since a loop space can be efficiently described as an operad algebra, it is not surprising that there are several existing definitions of n-category that utilize operad actions. These definitions fall into two main classes: those that define an n-category as an algebra of a higher order operad, and those that achieve an inductive definition using classical operads in symmetric monoidal categories to parameterize iterated enrichment. The first class of definitions is typified by Batanin and Leinster [5],[33].

The former author defines monoidal globular categories in which interchange transformations are isomorphisms and which thus resemble free strict n-categories. Globular operads live in these, and take all sorts of pasting diagrams as input types, as opposed to just a string of objects as in the case of classical operads. The binary composition in an n-category derives from the action of a certain one of these globular operads. Leinster expands this concept to describe n-categories with unbiased composition of any number of cells.

The second class of definitions is typified by the work of Trimble and May [52], [42]. The former parameterizes iterated enrichment with a series of operads in $(n-1)$-Cat achieved by taking the fundamental $(n-1)$-groupoid of the k^{th} component of the topological path composition operad E. The latter begins with an A_∞ operad in a symmetric monoidal category $\mathcal{V}$ and requires his enriched categories to be tensored over $\mathcal{V}$ so that the iterated enrichment always refers to the same original operad.

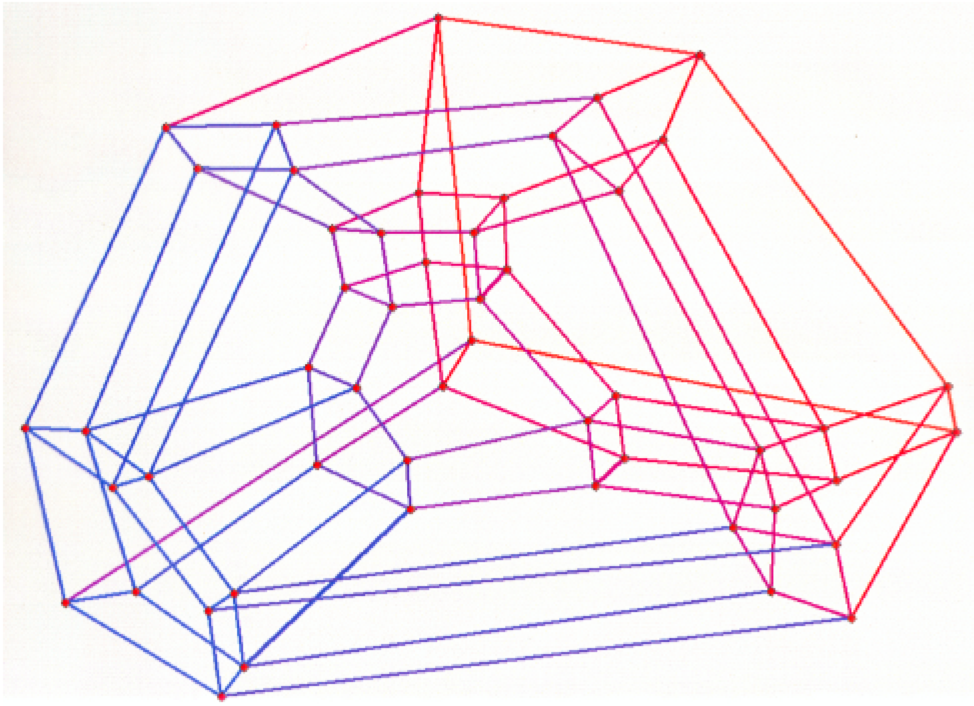


FIGURE 3. $\mathcal{C}K(5)$ as a convex hull.

Since (weak) enrichment over an operad algebra is described in terms of an action (on a graph over the algebra) of a left module over that operad, we plan to describe weakly enriched categories over each of these sorts of categories in those terms.

6.2. **Higher morphisms.** It has been shown by Hess et.al. that special operad bimodules called corings characterize functors between weak categories. Our polytopal examples demonstrate enriched functors as characterized by prisms of these bimodules. The prisms of course inherit the bimodule structure–they themselves form a bimodule. This fits with the results of [24]. This prism structure needs to be extended to the general case. Even less clear is how to characterize the higher morphisms. We hope for clues from the investigation of the polytopal case of naturahedra mentioned above.

6.3. **Enrichment, delooping, and modeling homotopy n-type k-fold loop spaces for $k > n$..** Recall that Batanin's groupoids have been shown to model homotopy types, and that these include loop spaces as special cases. Also recall that it is likely that if we wish to precisely model homotopy n-type k-fold loop spaces for $k > n$ then we may need to consider k-fold as well as k-tuply monoidal n-categories. In [19] are defined n-fold operads in n-fold monoidal categories in a way that is consistent with the spirit of Batanin's globular operads. Their potential value may include using them to weaken enrichment over n-fold monoidal categories in a way that is in the spirit of May and Trimble. This program carries with it the promise of characterizing k-fold loop spaces with homotopy n-type for all n, k by describing the categories with exactly those spaces as nerves. As a candidate for the type of category with such a nerve we suggest a weak n-category with k multiplications that interchange only in the lax sense. In this proposal "lax" will indicate that the morphisms involved in a definition are not necessarily isomorphisms. Lax interchangers will obey coherence axioms.

With either the existing theory or the proposed extension using n-fold operads, the ultimate investigation would be into the question of whether the proposed generalized enrichment is a good model of delooping. At this point the information may well flow the other way, as the discoveries about the nerves of weak n-categories inform as to the correct way to define enrichment.

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