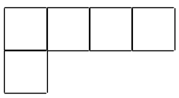
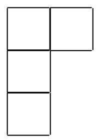
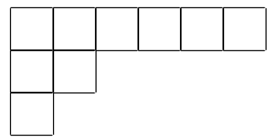
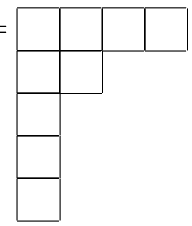
HIGHER PRODUCTS FOR YOUNG DIAGRAMS

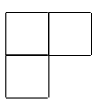
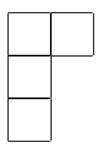
STEFAN FORCEY

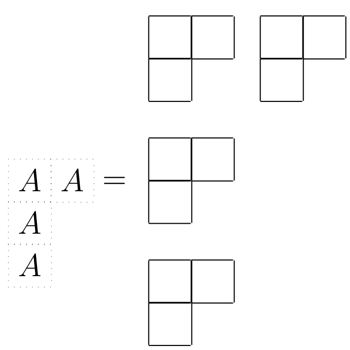
Recall our vertical and horizontal addition of Young diagrams. Let $\otimes_2$ be the product which adds the heights of columns of two diagrams, $\otimes_1$ adds the length of rows. We often refer to these as vertical and horizontal stacking respectively. If

$A =$  and $B =$  then $A \otimes_1 B =$  and $A \otimes_2 B =$ 

We can take as morphisms the totally ordered structure of the Young diagrams given by lexicographic ordering. The unit object is the zero diagram.

Here are some new products of Young diagrams, which also come in pairs. The first pair is "vertical and horizontal multiplication" and heuristically it consists of packing each box of diagram B with a copy of A. The two choices for multiplication then are the two ways of collapsing/shifting all the resulting boxes to form a new Young diagram (horizontally then vertically and vice versa). Of course sometimes there are more than these two choices, but it seems necessary to stick to the two basic choices in order to be well defined.

$A =$  and $B =$  then here is the heuristic packing:

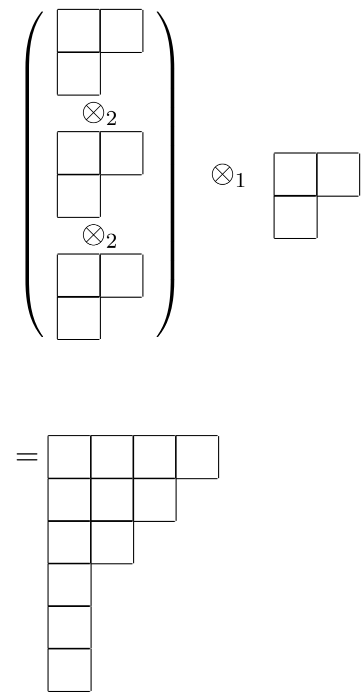


Key words and phrases. enriched categories, n-categories, iterated monoidal categories.

Thanks to XY-pic for the diagrams.

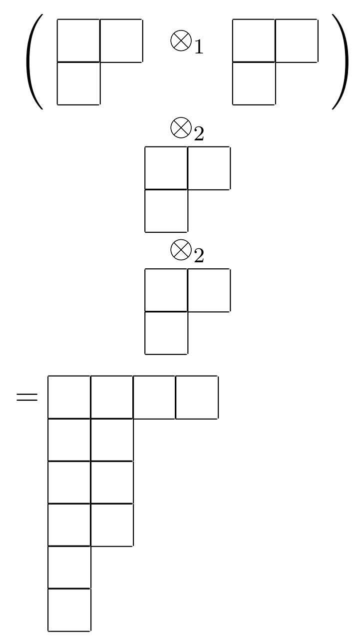
and so:

$A \boxtimes_1 B =$



and

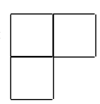
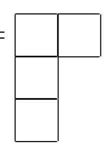
$A \boxtimes_2 B =$

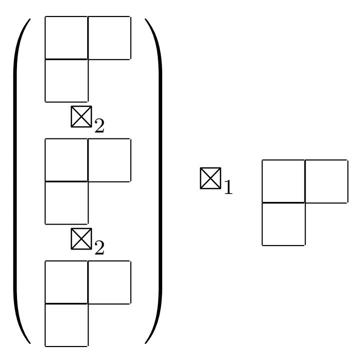


The unit object is the single box diagram. The 2-fold monoidal structure is the existence of interchange morphisms $A \boxtimes_2 B \boxtimes_1 C \boxtimes_2 D \to A \boxtimes_1 C \boxtimes_2 B \boxtimes_1 D$. Of course we need to show associativity as well. Does this structure make the Young diagrams into a 2-fold rig (ring without negatives) category, or if we invent negatives, an actual 2-fold ring? What are even the axioms of a 2-fold ring category? 2 additions, 2 multiplications, 2 units, distributivity of respective operations, interchange relations, and perhaps some sort of hybrid distributivity.

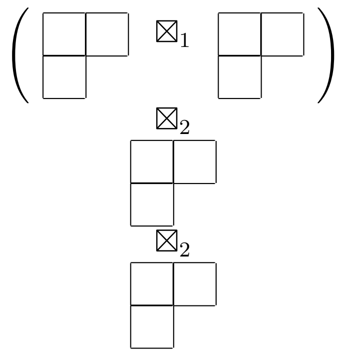
Note that these multiplications are commutative. Thus we can immediately generalize the multiplication to operadic structures. The object $\mathcal{C}(n)$ of the operad is the set of n-box Young diagrams. There is a function $\mathcal{C}(n) \times \mathcal{C}(j_1) \times \cdots \times \mathcal{C}(j_n) \to \mathcal{C}(\sum j_i)$ once we have described in what order to pack the $\mathcal{C}(j_i)$ into the boxes of $\mathcal{C}(n)$ and chosen a canonical order in which to collapse the result–i.e. either horizontally then vertically or vice versa. Say we always pack in the same order (rows first? column first? diagonally?) and so given that order there are two compositions taking us to $\mathcal{C}(\sum j_i)$, γ_1 given by vertical collapse first and γ_2 given by horizontal collapse first. Thus γ_2 gives us a taller result as usual. Can anything else be said about the interaction of these two compositions? What is the effect of the chosen packing order?

The iteration of this construction now proceeds apace. Let the next pair of products be called "vertical and horizontal exponentiation." Then for the same A, B as immediately above:

$A =$  and $B =$  then

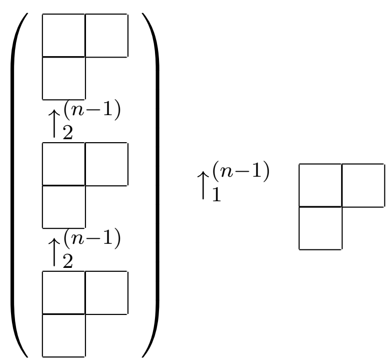
$A \wedge_1 B =$ 

and

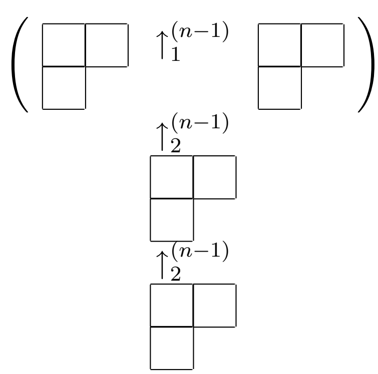
$A \wedge_2 B =$ 

Note that exponentiation is not commutative, and likely not associative (use right-associativity, i.e. evaluate from the right). Questions: do normal rules of exponents apply? Is there an associator if we put in parentheses? Is there a 2-fold monoidal structure?

The general case given by up-arrow notation, where $\wedge_i = \uparrow_i$ and $\uparrow_i^n = \uparrow_i \dots \uparrow_i$ with n arrows:

$A \uparrow_1^n B =$ 

and

$A \uparrow_2^n B =$ 

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