1. Project description: Geometric combinatorial Hopf algebras: Summary

This proposal is about combinatorial algebra, with a geometrical flavor. Together with students and collaborators I have been developing a diverse family of graded algebras and coalgebras, modules and comodules, often with Hopf and differential structures. Their common feature is that all are based upon *recursive sequences of convex polytopes.* The key point of interest in each case is to see how the algebraic structure reflects the combinatorial structure, and vice versa. We are building upon the foundations laid by many other researchers, especially Gian-Carlo Rota, who most clearly saw the strength of this approach. The historical examples of Hopf algebras $\mathfrak{S}Sym$ and $\mathcal{Q}Sym$, the Malvenuto-Reutenauer Hopf algebra and the quasisymmetric functions, can be defined using graded bases of permutations and boolean subsets respectively. Loday and Ronco used the fact that certain binary trees can represent both sorts of combinatorial objects to discover the Hopf algebra $\mathcal{Y}Sym$ lying between them. Chapoton capitalized on the fact that the three graded bases could actually be described as the vertex sets of polytope sequences, and defined larger algebras on the faces of the permutohedra, associahedra and cubes.

The polytope sequences we study include those familiar examples as well as newer families such as the graph multiplihedra and composihedra. Simultaneously with our study of Hopf algebras we have been developing entire new classes of combinatorially defined polytopes, both for their own considerable interest and as fresh examples upon which to test our algebraic hypotheses.

The individual polytopes often possess lattice structures; the sequences in their entirety possess operad and operad module structures, and whole families of sequences are related by cellular surjections and lattice quotients. But which polytopes are guaranteed to have recursive properties and lattice structure, what maps are guaranteed to exist, and when do the geometric combinatorial properties guarantee graded algebra structures? We already know a good deal, but there are more open questions than any one of us has time to address! Luckily the questions are perfect for introducing students to pure math research and experimentation. Already several students have been busy building examples of our new polytopes in low dimension, looking for new patterns and testing conjectures. Others have also been indispensable in tirelessly checking the properties of proposed algebraic structures, and finding proofs in the process.

The proposed work can be divided into a series of expanding classes of associahedron-like polytope sequences. We begin with the most familiar, defined using trees. These are generalized to a broader family described by simple graphs, then further to include multigraph based polytopes, and finally to the level of polytopes determined by CW-complexes. There are several famous quotient maps between sets of trees. As the definition of associahedron broadens, there are revealed finer filtrations and further extension of these maps. The importance of the maps is that they yield corresponding algebraic homomorphisms between the Hopf algebras and modules built upon the face posets of each polytope sequence.

At each level of the expansion, we plan a tangent investigation into a derived collection of polytope sequences. The central character here is the multiplihedron, originally developed as the parameter space for describing maps between homotopy-associative H-spaces. For each sequence of associahedra we plan to study there is a collection of nine interesting polytope sequences that are related to the corresponding multiplihedra via cellular surjections of polytopes. For each of the nine sequences there are two ways of constructing graded algebras, coalgebras or Hopf algebras. Again the polytope projections turn out to underlie algebra projections.

2. Introduction

A combinatorial sequence that is created by a recursive process often carries the seed of a graded algebraic structure. An algebra reflects the process of building a new object from prior ones; and a coalgebra arises from deconstructing an object into its constituent components.

The discrete geometric approach we are taking is to study recursive combinatorial structures among polytopes and polytopal cones. There is a vast landscape of these shapes which have inductively defined facets. The challenge we face is to pick out from this space the sequences which give rise to interesting algebras and bialgebras.

In the remainder of this proposal I will give details about our current knowledge, conjectures and the most pressing questions we wish to answer. Throughout there are the two themes of combinatorial polytopes and Hopf algebras they support. We choose to organize the proposal by the polytopes, interweaving the algebra as dictated by the geometry.

We begin with the unified description of associahedra and permutohedra as graph-associahedra developed by Carr and Devadoss. Our first new insight is to use that point of view to describe the algebra and coalgebra structures of $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$ in completely geometric terms, using cartesian products, inclusion maps, and cellular surjections of the polytopes. The polytope maps give rise to algebra and coalgebra homomorphisms. These in turn allow us to describe new algebraic structures, based on the vertex sets and on the faces of special sequences of graph associahedra, which are quotients of $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$. These include algebras based upon the cyclohedra (cycle graphs), the simplices (edgeless graphs), and upon several infinite sequences of polytope families which filter the historical algebras and coalgebras. The simplices are especially interesting, since their face Hopf algebra $\Delta\tilde{Sym}$ has the same graded dimension as $\mathcal{Q}Sym$. This implies a unique coalgebra map between the two, since Aguiar, Bergeron and Sottile have shown $\mathcal{Q}Sym$ to be terminal [1].

Next we plan to continue our study of a new algebra and Hopf module structure on the classic multiplihedra, seen as the lattice of bi-leveled trees, which was begun in [13], with Aaron Lauve and Frank Sottile. The algebra of bi-leveled trees allows an algebra homomorphism from $\mathfrak{S}Sym$. We plan to carry this approach further to look at quotients of both the permutohedra and multiplihedra described by selectively and progressively forgetting distinguishing structure of the bi-leveled trees, alternately in the upper and lower levels. First we ignore the lengths of internal branches, then the pattern of internal branching itself. The level corresponds to the image of 1 in the permutation, or the level of attachment of the leftmost branch. The midpoint of these cellular surjections of polytopes is the classic multiplihedra. The endpoint is the cubes.

By alternatively representing all the vertices of all the polytopes in our big commuting diamond by painted binary trees, where the paint shows the level rather than the left-most leaf, we are able to describe new Hopf algebras on each of the nine combinatorial sequences. We already have initial results from this viewpoint, including finding the antipodes and primitives of painted binary trees, which we believe will generalize to the whole picture. The new Hopf algebra on the vertices of the cubes, with vertices represented as painted trees with all branching structure forgotten both above and below the paint-line, is dual to our Hopf algebra of simplex faces.

For every graph there is a polytope called the graph multiplihedron whose face poset is isomorphic to a poset of marked tubings, which are the analog of painted trees. Thus for every sequence of graphs that underlies a new sub-algebra or sub-coalgebra of $\mathfrak{S}Sym$, there is a new big diamond of nine polytope sequences that support a whole family of related algebras and coalgebras. The permutohedra and the cubes respectively begin and end each commuting sequence of diamonds.

Since the projection from permutohedra to multiplihedra factors through all connected graph multiplihedra, we expect algebra structures on each of the new polytope sequences.

With Satyan Devadoss and Mike Carr, I have recently begun preparing research on a whole new class of polytopes with recursive structure; the *multigraph associahedra*. Since the recursive combinatorial geometric structure always underlies the algebra, our first task is to find inductive descriptions of the facets. Geometric realizations using truncations and convex hulls are the key to unlocking these secrets; especially since they allow experimentation in higher dimensions with computers.

The generalization of associahedron extends even further to CW-complexes. I have isolated sequences of the new multigraph and CW-complex associahedra which do support new Hopf algebras. There are several which may relate directly to product algebras of $\mathcal{Q}Sym$, $\mathcal{Y}Sym$ and $\mathfrak{S}Sym$. There are also some new sequences that support coalgebras. The cubes again appear as a sequence of CW-complex associahedra corresponding to globular complexes of increasing dimension. There are even further steps in the future, where we plan to investigate connections to the generalized associahedra of Fomin and Zelevinsky, as well as the generalized permutohedra of Postnikov.

3. Polytopes in Algebra

3.1. **Review of important Hopf algebras based on trees.** In 1998 Loday and Ronco found an intriguing Hopf algebra of planar binary trees lying between the Malvenuto-Reutenauer Hopf algebra of permutations [18] and the Solomon descent algebra of Boolean subsets [17]. They also described natural Hopf algebra maps which neatly factor the descent map from permutations to Boolean subsets. Their first factor turns out to be the restriction (to vertices) of the Tonks projection from the permutohedron to the associahedron. Chapoton made sense of this latter fact when he found larger Hopf algebras based on the faces of the respective polytopes [6].

Much more of the structure of these algebras has been uncovered in the last decade. In 2005 and 2006 Aguiar and Sottile used alternate bases for the Loday-Ronco Hopf algebra and its dual to construct explicit isomorphisms [3],[2]. Several descriptions of the big picture of combinatorial Hopf algebras have put these structures in perspective, notably [1], and [15], and most recently the preprint of Loday and Ronco [16].

3.2. **New insights into $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$.** In our recent (submitted) paper [14] we use a new point of view, graph associahedra, to study these algebras. First we show how the Hopf algebras $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$ and the face algebras $\mathfrak{S}\tilde{S}ym$ and $\mathcal{Y}\tilde{S}ym$ containing them can be understood in a unified geometrical way, via cellular surjections and recursive facet inclusion. For an example of the product of basis elements of $\mathfrak{S}Sym$, see Figure 1. Then we capitalize on that unified viewpoint to build analogous algebraic structures on the vertices and faces of the cyclohedra and simplices, which we describe below in Section 5.1.

3.3. **Cambrian lattices and their Hopf algebras.** In [23] Reading uses *insertional* and *translational* lattice congruences to build the Malvenuto-Reutenauer Hopf algebra as the limit of a sequence of smaller Hopf algebras: the first Hopf algebra in the sequence is the graded Hopf algebra with one-dimensional graded pieces and the second is the Hopf algebra of non-commutative symmetric functions. In the same paper Reading builds the Hopf algebra of planar binary trees as the limit of a similar sequence. Our work here is partly motivated by the question of whether the third leg of the triangle might also be amenable to such an approach. As an answer, in Sections 5.2 and 5.3 we build the Malvenuto-Reutenauer Hopf algebra as the limit of a sequence of smaller graded algebras where the first algebra in the sequence is the graded algebra of planar binary trees, and the second is based upon the cyclohedra.

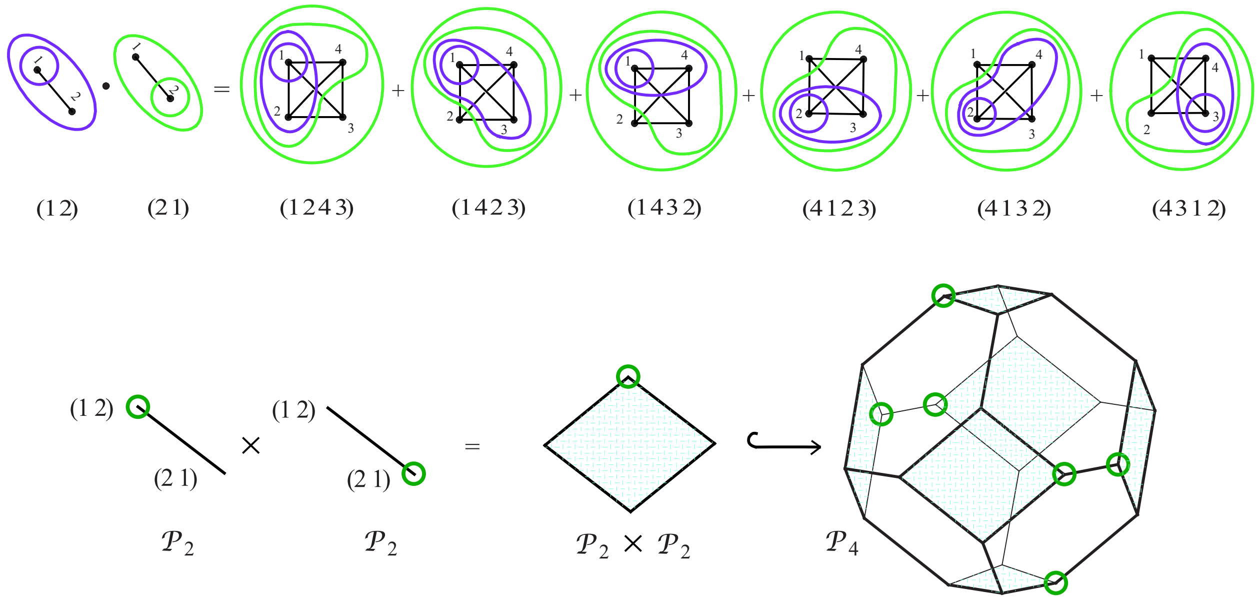
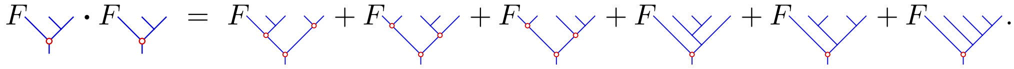


FIGURE 1. The theme of [14] is that the product of two faces, here from terms $\mathcal{P}_i$ and $\mathcal{P}_j$ of a given recursive sequence of polytopes, is described as a sum of faces of the term $\mathcal{P}_{i+j}$. The summed faces in the product are the images of maps which embed cartesian products of earlier terms of $\{\mathcal{P}_n\}$.

4. Multiplihedra: trees

4.1. **The classic polytopes part 1: bileveled trees.** The vertices of the multiplihedra can be represented by trees whose nodes are divided into two levels. In [13] and [12] we have demonstrated a new graded algebra with basis these bileveled trees, graded by the numbers of internal nodes. Here is an example of the product:



The important role played by the new algebra, which we call $\mathcal{M}Sym$, is due to the fact that the Saneblidze-Umble cellular projection β from the permutohedra to the multiplihedra (restricted to vertices) is indeed an algebra homomorphism in this context. Thus we are led to ask:

Question 4.1. *Is there an extension of $\mathcal{M}Sym$ to the faces of the multiplihedra paralleling Chapoton's extension of $\mathfrak{S}Sym$ to the faces of the permutohedra? If so, does it also have a differential graded structure?*

We conjecture that the answer is yes, but a precursor to this question is the mystery of the geometric structure of the product in $\mathcal{M}Sym$. It can be observed from the cellular projection β, but the precise description has not been worked out yet.

4.2. **A big commuting diagram of bileveled trees.** In [10] I introduced a new sequence of polytopes which are simultaneously quotients of the multiplihedra and categorified versions of the associahedra. The former is seen by forgetting tree structure in the upper portion of a bi-leveled tree; the latter is due to the fact that these new *composihedra* parameterize enrichment of categories. In fact, several authors had in the past assumed that the composihedra *were* the associahedra. Their mistake came in part due to the fact that the associahedra are indeed the result of forgetting tree structure of the lower level of the bi-leveled trees.

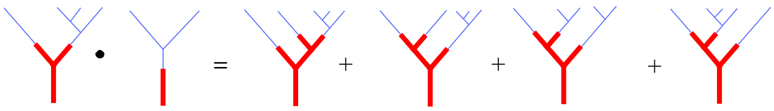
Since that paper, we have realized several additional polytope sequences exist whose terms' vertices correspond to forgetting ordering and or tree structure of the two levels at different times.

When upper and lower tree structure is forgotten, what remains is a picture of a composition, that is, a vertex of a cube. Figure 2 is a pair of commuting diagrams, one showing examples of an ordered tree and all its simpler images, and the second showing the corresponding polytopes in 3d.

Each sort of tree promises to be a basis element of a new graded algebra; subalgebras of $\mathfrak{S}Sym$ which are either containing or contained within $\mathcal{M}Sym$. The main question that is begging for attention here is:

Question 4.2. *What relationships exist between the classic Hopf algebras $\mathcal{Y}Sym$ and $\mathcal{Q}Sym$ and our new algebras that use the same bases?*

4.3. **The classic polytopes part 2: painted trees.** In the process of writing the recent papers [13] and [12] we realized the existence of a new Hopf algebra based upon the vertices of the multiplihedra, but described via the pictures of painted trees. Here is an example of the product:



The coproduct is the usual splitting of trees. The antipode is described in our forthcoming paper [11].

In [3] the authors use Möbius inversion on the Tamari lattice of binary trees to describe the primitive elements of $\mathcal{Y}Sym$. There is some overlap with our new algebras, in that a primitive of $\mathcal{Y}Sym$ clearly gives rise to several painted primitives.

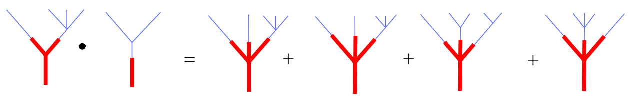
Question 4.3. *We would like to find a nice description of the primitive elements of the painted trees. Is there a sublattice of the multiplihedron whose Möbius inversion will provide a simple characterization of the space of primitives?*

We conjecture that the answer is yes. Frank Sottile is fairly certain that the proof he has found will generalize to an entire new family of Hopf algebras which we will discuss next.

4.4. **The big commuting diamond of painted trees.** The bi-leveled trees of Figure 2 all have alternate representations, using painted tree versions with one less leaf. Figure 3 is the picture of the painted trees corresponding precisely to the ones in the same positions as in Figure 2.

Every one of these sorts of trees provides the basis for a graded Hopf algebra.Here is the new sort of product, using corollas rather than combs, and demonstrated on compositions.

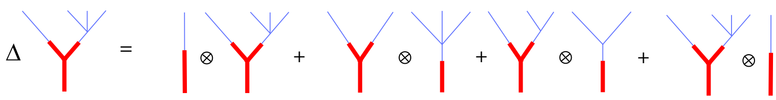
$F_{13} \bullet F_2 =$



$$F_{113} + F_{113} + F_{122} + F_{131}$$

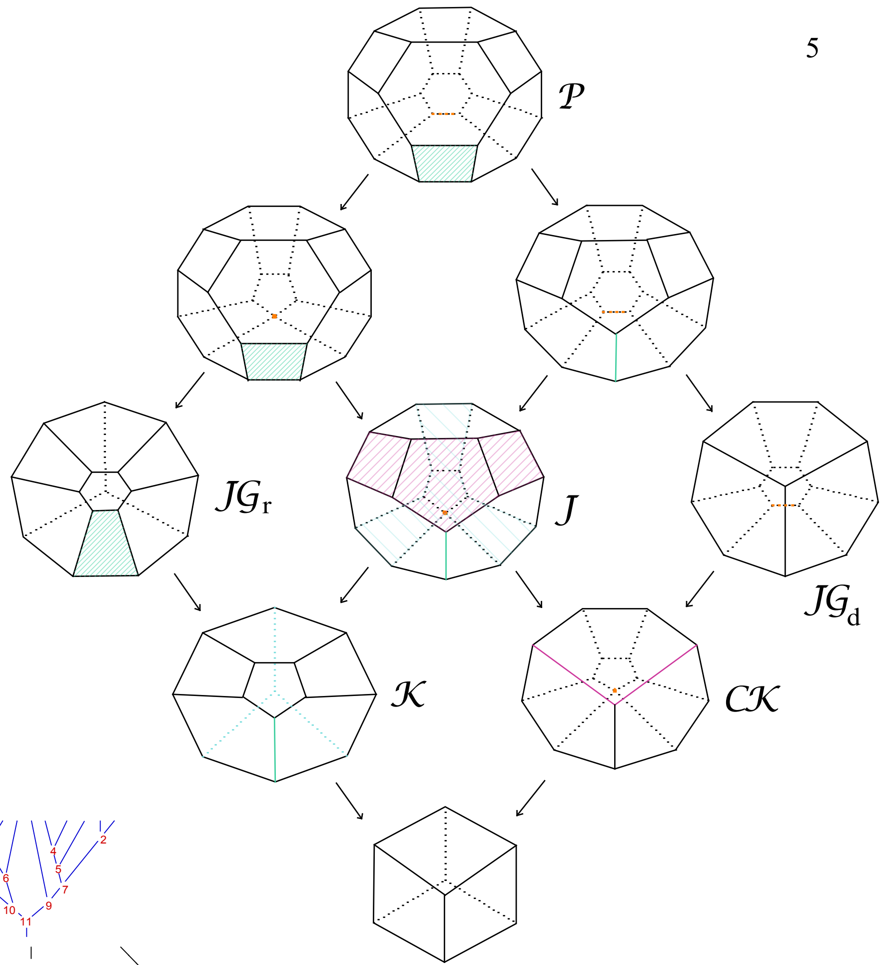
The coproduct is the usual splitting of trees:

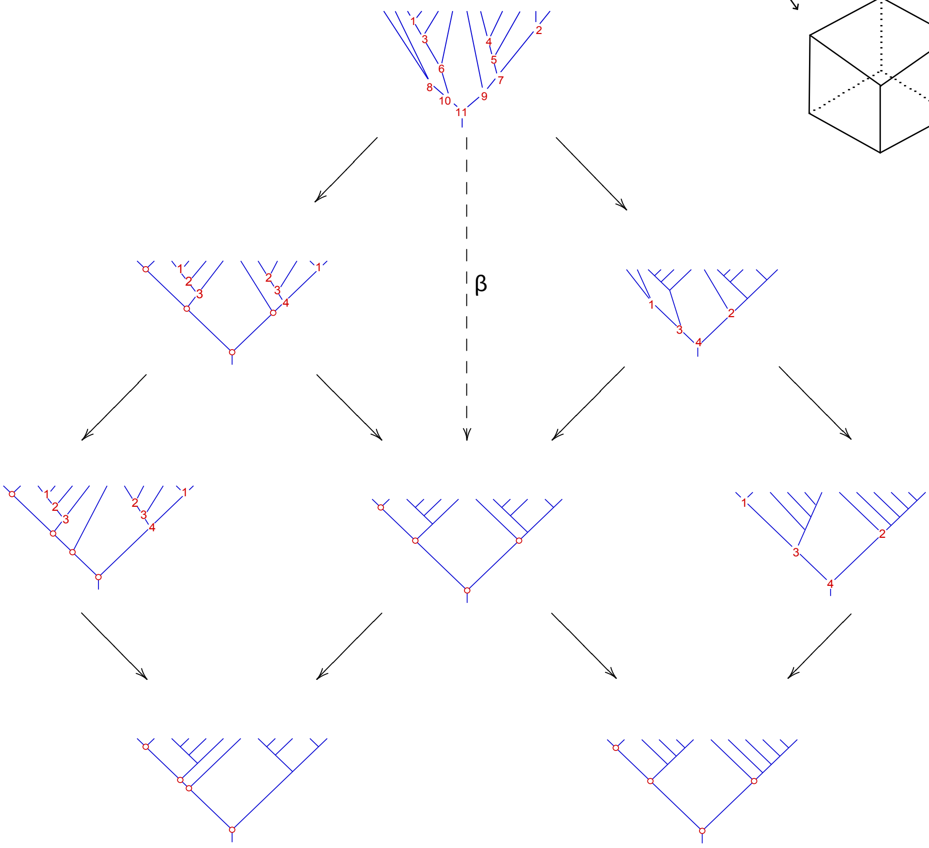
$\Delta\, F_{13} =$



$$= F_1 \otimes F_{13} + F_{11} \otimes F_3 + F_{12} \otimes F_2 + F_{13} \otimes F_1$$

This implies all new Hopf algebra structures on the familiar polytopes: permutohedra, associahedra and cubes. The question is even more important since we are comparing Hopf algebras:





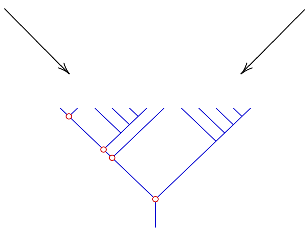


Figure 2. On the top is a commuting diagram of polytopes in 3 dimensions, from the permutohedron to the cube. The multiplihedron is in the middle, $\mathcal{K}$ is the associahedron, and $\mathcal{CK}$ is the composihedron. Below that are trees corresponding to vertices of those polytopes (in ten dimensions) showing the action of the surjections.

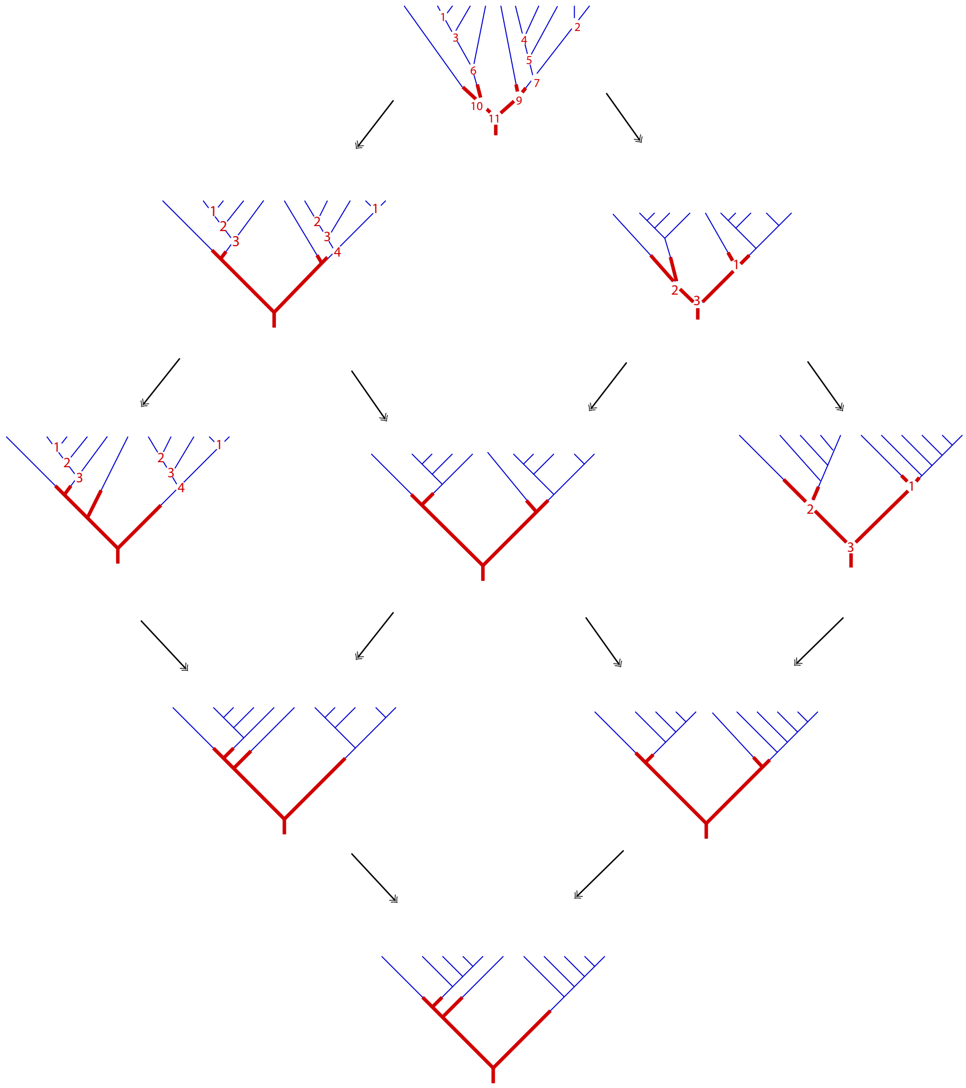


Figure 3. These trees correspond to the same vertices of polytopes (in ten dimensions) as the trees shown in Figure 2, but each of these allow us to describe Hopf algebras.

Question 4.4. *How are the new Hopf algebra structures related to $\mathfrak{S}Sym$, $\mathcal{Y}Sym$ and $\mathcal{Q}Sym$? Note especially that $\mathcal{Q}Sym$ has been shown to be terminal in the category of coalgebras. Thus there should be a unique map from our new Hopf algebra to $\mathcal{Q}Sym$. Is this map an isomorphism?*

5. Graph associahedra and cellular projections.

In [4] and [8] Carr and Devadoss show that for every graph G there is a unique convex polytope $\mathcal{K}_G$ whose facets correspond to connected induced subgraphs. I first suspected the existence of new algebras based on $\mathcal{K}_G$ after my discovery (published in [14]) that the Tonks projection from the permutohedron to the associahedron can be factored through a series of graph-associahedra. This fact is simple to demonstrate; it follows from Devadoss's discovery that the complete graph-associahedron is the permutohedron while the path graph-associahedron is the Stasheff polytope. Thus by deleting edges of the complete graph one at a time, we describe a family of quotient cellular projections. Figure 4 shows one of these. Our important result is that the cellular projections give rise to graded algebra maps.

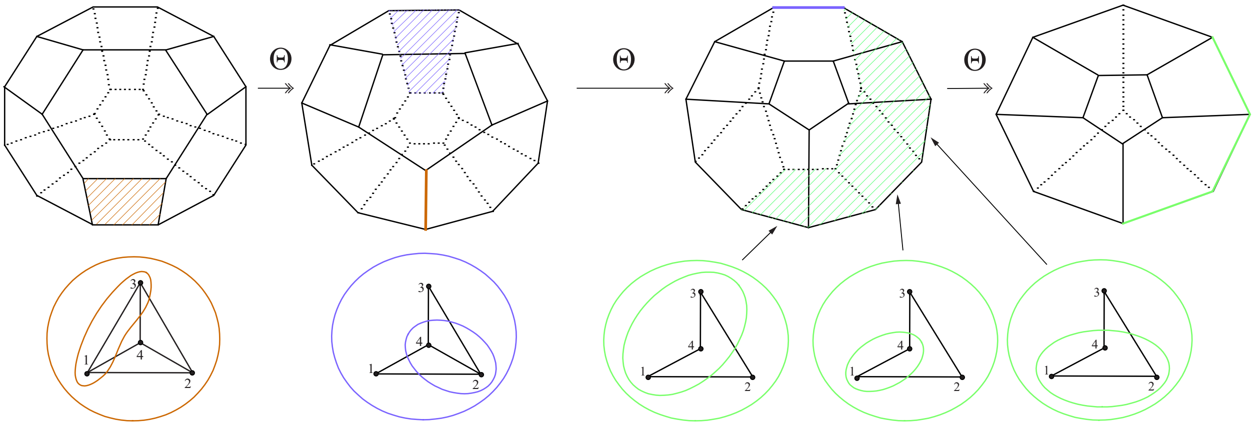


FIGURE 4. A factorization of the Tonks projection through 3 dimensional graph associahedra. The shaded facets correspond to the shown tubings, and are collapsed as indicated to respective edges. The first, third and fourth pictured polytopes are above views of $\mathcal{P}_4, \mathcal{W}_4$ and $\mathcal{K}_4$ respectively.

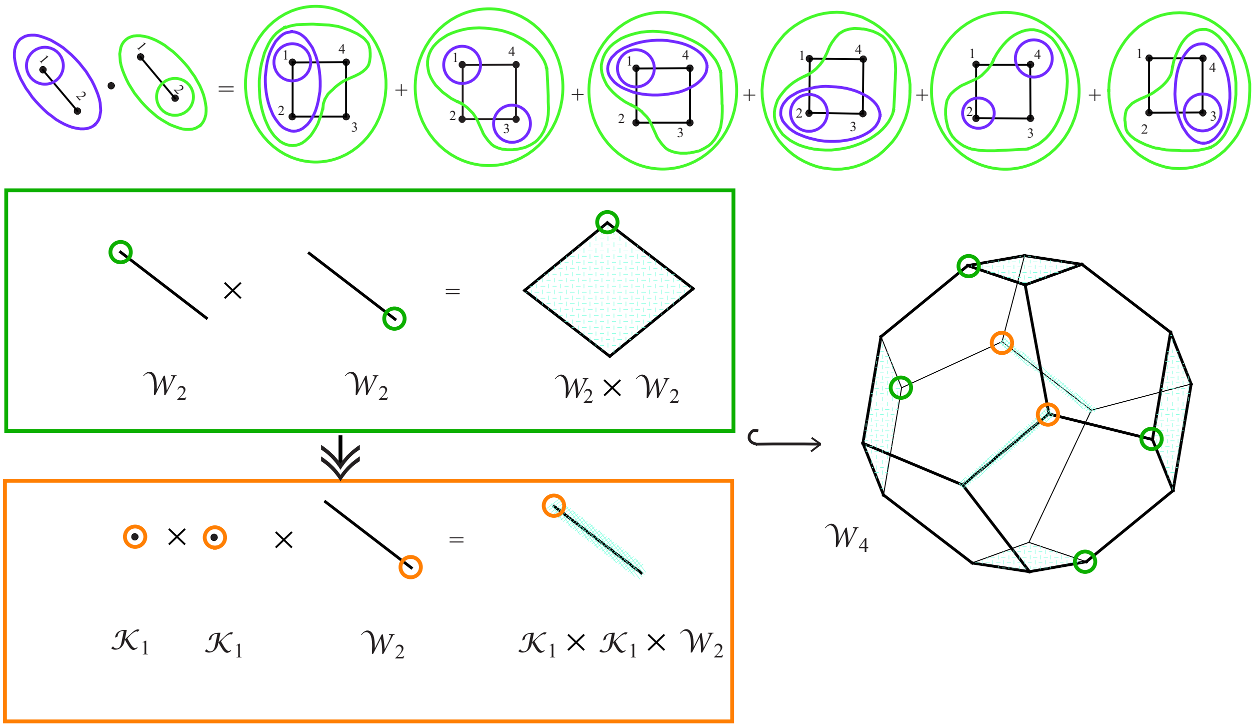


FIGURE 5. The theme of [14] is that the product of two faces, here from terms $\mathcal{W}_i$ and $\mathcal{W}_j$ of the cyclohedra, is described as a sum of faces of the term $\mathcal{W}_{i+j}$. The summed faces in the product are the images of maps which embed cartesian products of earlier terms of $\{\mathcal{W}_n\}$, composed with our new extensions of the Tonks projection.

5.1. **New algebras and modules: cyclohedron and simplex.** In our recent paper [14] we demonstrate associative graded algebra structures on the vertices of the cyclohedra and simplices, denoted $\mathcal{W}Sym$ and ΔSym. We also extend this structure to the full poset of faces. Figure 5 shows the product of a pair of vertices in the cyclohedron.

The number of faces of the n-simplex, including the null face and the n-dimensional face, is 2^n. By adjoining the null face here we thus have a graded algebra with n^{th} component of dimension 2^n. A fascinating convergence now appears.

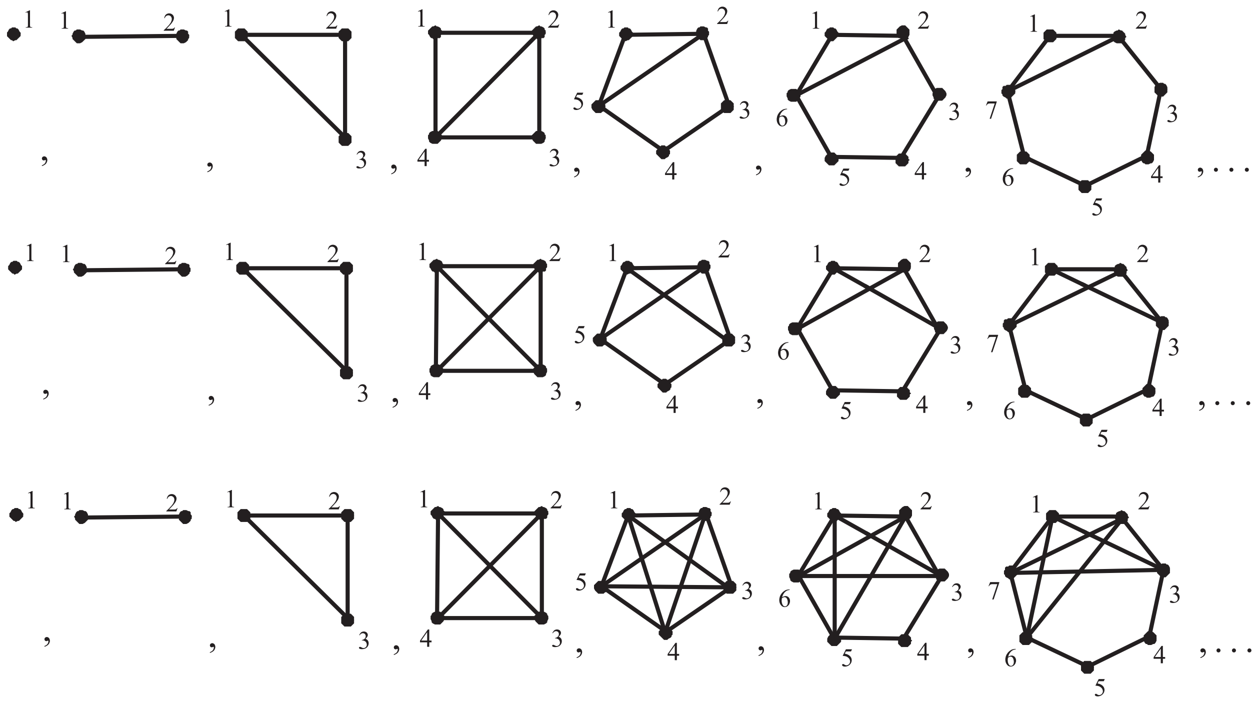
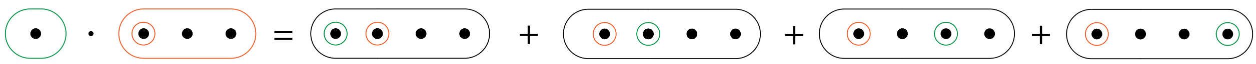


FIGURE 6. Hereditary graph sequences.

Conjecture 5.1. *The Hopf algebra of simplex faces $\Delta\tilde{S}ym$ is dual to the algebra of compositions with the painted tree product.*

Here is an example; compare this with the example in section 4.4 by using the bijection between subsets $\{a, b, \dots c\} \subset [n]$ and compositions $(a, b-a, \dots, n+1-c)$ of $n+1$.

$$F_{\emptyset} \bullet F_{\{1\}} =$$



$$= F_{\{1,2\}} + F_{\{1,2\}} + F_{\{1,3\}} + F_{\{1,4\}}$$

5.2. **More new algebras: filtering $\mathfrak{S}Sym$.** Next is a description of how to construct a graded algebra which lies between $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$. First we need to define a sequence of graphs with numbered nodes, indexed by the number of nodes in each graph. The recursive property which they minimally must possess is as follows:

Definition 5.2 (**Hereditary graph sequence**). *Given a graph G_n in our sequence, and any one of its nodes: the reconnected complement of G_n with that node deleted and with the inherited numbering of nodes must have as a subgraph the term G_{n-1} of our sequence. Figure 6 shows several examples.*

Conjecture 5.3. *The vertices of a polytope sequence $\mathcal{K}_{G_i}$ for $\{G_i\}$ a hereditary graph sequence form the basis of a subalgebra of $\mathfrak{S}Sym$.*

The product of two basis elements is performed in an analogous way to the product in $\mathfrak{S}Sym$ (and $\mathcal{W}Sym$). Figure 7 shows a partial example. The definition of hereditary graph sequence is designed to ensure that the product is well-defined. We have examples of this conjecture in

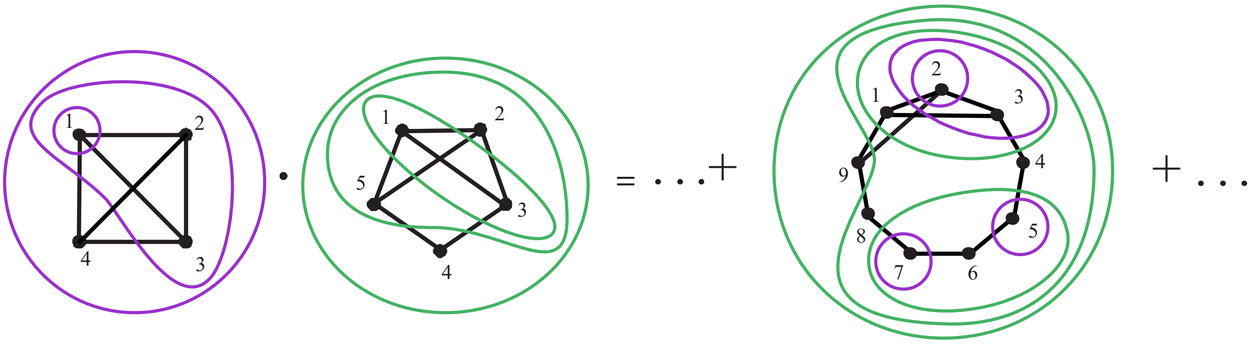


FIGURE 7. One term in the indicated product.

the form of $\mathcal{Y}Sym$, $\mathcal{W}Sym$ and ΔSym. The proof of associativity rests upon a straightforward demonstration of the following fact:

Conjecture 5.4. *The generalized Tonks projection from the permutohedra to a sequence $\mathcal{K}_{G_i}$ for a hereditary graph sequence is an algebra homomorphism.*

There is a lot to investigate here. From the examples in Figure 6 we conjecture that there is an infinite filtration of the algebra $\mathfrak{S}Sym$ based on incrementally increasing the connectedness of the early graphs in the hereditary sequence. The first algebra in the filtration is $\mathcal{Y}Sym$ and the second is $\mathcal{W}Sym$.

5.3. **New Coalgebras: another filter of $\mathfrak{S}Sym$.** We consider another class of sequences of graphs where G_n has numbered nodes $\nu_1, \dots, \nu_n$.

Definition 5.5 (**Restrictive graph sequence**). *Given a graph G_n in our sequence: the induced subgraph of $G - \nu_n$ and the induced subgraph of $G - \nu_1$ both equal the term G_{n-1} of our sequence. Figure 5.3 shows examples.*

Conjecture 5.6. *The vertices of a polytope sequence $\mathcal{K}_{G_i}$ for $\{G_i\}$ a restrictive graph sequence form the basis of a subcoalgebra of $\mathfrak{S}Sym$.*

The coproduct of a basis element is performed in an analogous way to the coproduct in $\mathfrak{S}Sym$. Figure 9 shows an example. The definition of restrictive graph sequence is designed to ensure that the coproduct is well-defined. The proof of coassociativity rests upon a straightforward demonstration of the following fact:

Conjecture 5.7. *The generalized Tonks projection from the permutohedra to a sequence $\mathcal{K}_{G_i}$ for a restrictive graph sequence is a coalgebra map.*

From the examples in Figure 5.3 we conjecture that there is an infinite filtration of the coalgebra $\mathfrak{S}Sym$ based on incrementally increasing the connectedness of the early graphs in the restrictive sequence. Again the first coalgebra in the filtration is $\mathcal{Y}Sym$.

5.4. **Face algebras.** As usual, we conjecture that the algebras and coalgebras based upon the graph associahedra for hereditary and restrictive graph sequences may be enlarged to respective structures on the faces of the polytopes.

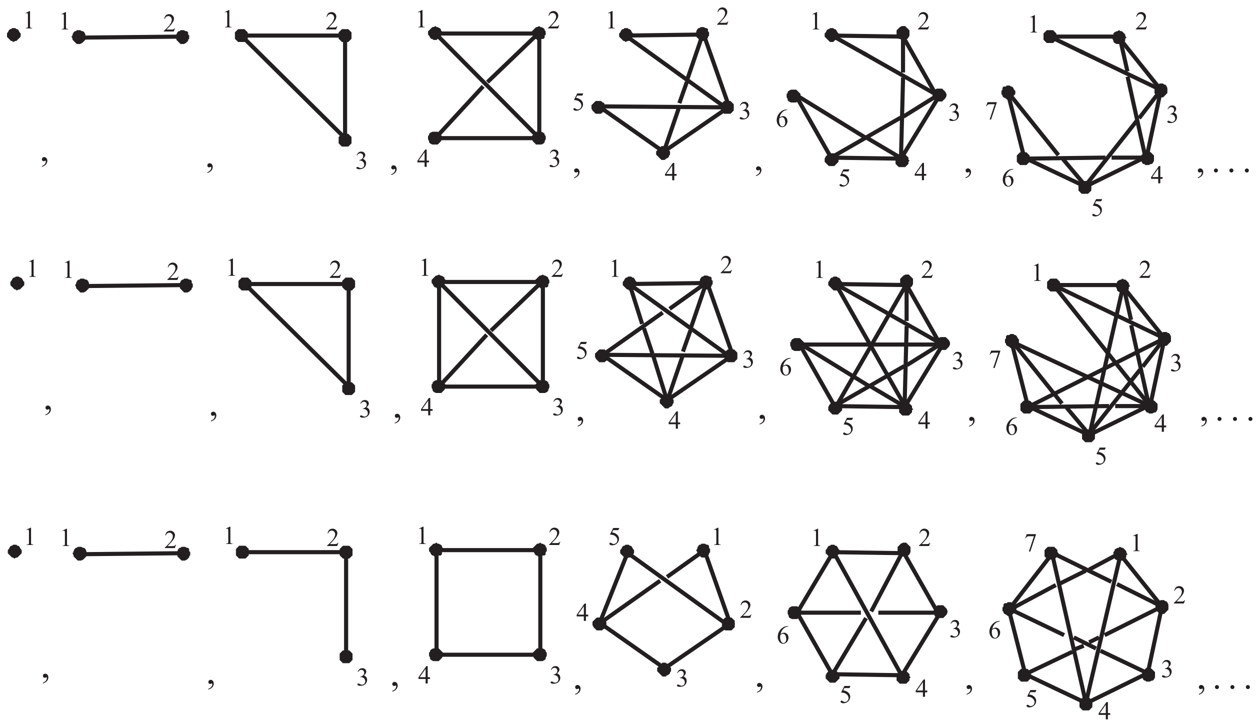


FIGURE 8. Restrictive graph sequences.

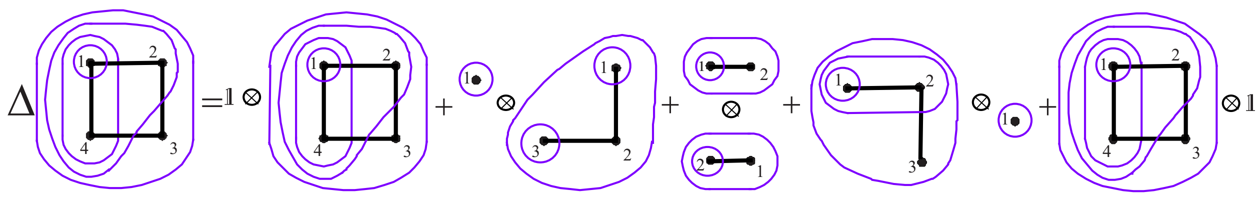


FIGURE 9. A coproduct in a restrictive sequence coalgebra.

5.5. **Investigative strategies.** We have used several basic principles in order to discover new algebra structure. One is the existence of cellular surjections of polytopes, which allow a structure on a polytope sequence to be reflected in the projected image. Another is the recursive nature of the polytope sequences, which allow products and coproducts as well as actions and coactions to be formulated in terms of cartesian products. A third is simply by finding analogies between the combinatorial sets that index the faces of our polytopes, and mirroring these in the definition of new operations. Further methods we hope to develop include the use of operads and operad modules, the exploitation of various lattice quotients, and the specialization of the theory of species to our setting.

5.5.1. *Lattices.* One can represent the elements of the symmetric groups in multiple ways. Classically these pictures have allowed lattices such as the Tamari order on binary trees and the Boolean posets to be seen as projections of the weak order on symmetric groups. We have uncovered several new poset structures on the skeletons of the graph associahedra. Given a numbering of the nodes of a graph we define the *ordered graph lattice.* We can describe the conjectural covering relations as follows: a maximal tubing covers another if the collection of all the tubes of both splits into identical pairs except for one pair of tubes which are unique to their respective tubings. Ordering the numbered nodes in the two tubes which make up this pair, the greater tubing is the one whose unique tube has a lexicographically greater list of nodes. This ordering generalizes both the weak order on permutations and the Tamari ordering of binary trees. Figure 10 shows an example.

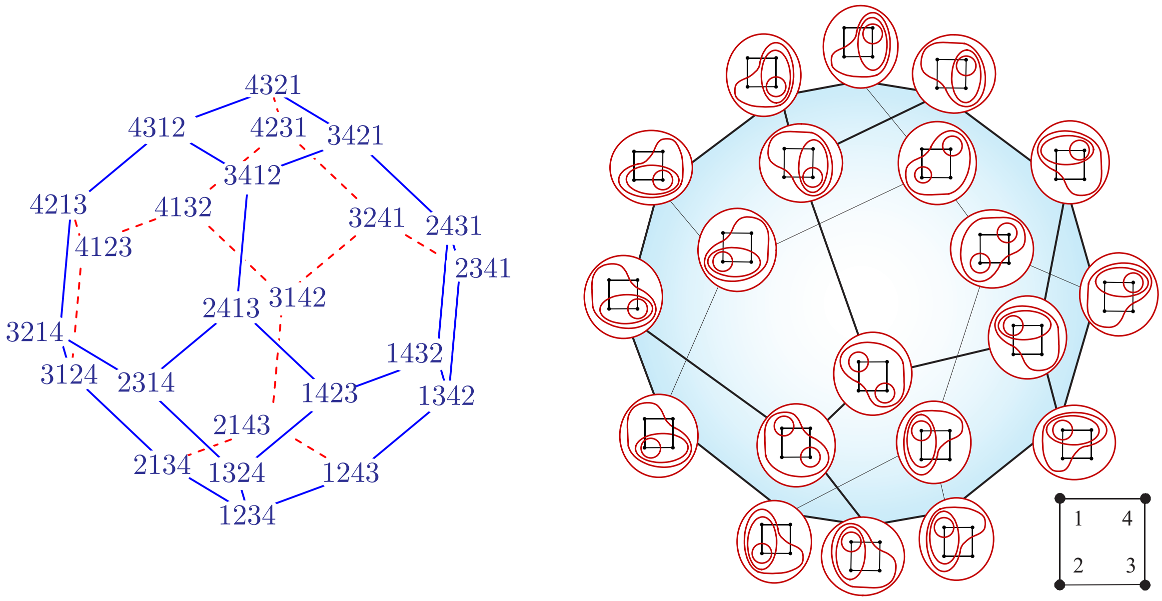


FIGURE 10. On the left is the 1-skeleton of the permutohedron demonstrating the weak order on $\mathfrak{S}_4$. On the right is the 1-skeleton of the cyclohedron, seen as a lattice of graph tubings. Möbius inversion on this lattice gives a new basis for the corresponding noncommutative graded algebra. Notice that the new poset structure is different than Reading's Cambrian lattice on the same set [24].

Conjecture 5.8. *The 1-skeleton of each graph associahedron is a quotient lattice of the weak order on the symmetric group.*

Of course the projection map we have in mind is the generalized Tonks projection restricted to vertices. The answer to the following question will have important ramifications to finding Möbius inversion.

Question 5.9. *Do the generalized Tonks projections from $\mathfrak{S}_n$ to our new lattices form lattice congruences or interval retracts?*

5.5.2. *Möbius inversion.* By performing Möbius inversion on the elements of our basis, with respect to the various lattices, we can find a new basis for the algebra or coalgebra. As mentioned above, this new basis can prove to be perfect for describing primitives or finding structure constants. In [12] we show that the existence of an interval retract between our lattice and the weak order on $\mathfrak{S}_n$ implies a formula for the Möbius function.

5.5.3. *Generating functions.* Generating functions have recently been discovered for many of the combinatorial sequences involved in our research. In [21] Postnikov, Reiner and Williams uncovered the generating functions for the vertices of many graph types, including cycles. I found the generating functions for the vertices of the multiplihedra and composihedra in [9] and [10]. These are useful both for finding dimension by counting basis elements and for existence theorems based on the fundamental theorem of Hopf modules.

5.5.4. *Operads and operad modules.* The recent work of Chapoton and Livernet stands out; their paper [5] shows that the commutative Connes-Kreimer Hopf algebra of rooted trees is in fact the incidence algebra based upon the operad of rooted trees, as defined by Schmitt in [26]. Furthermore

Chapoton points out that this incidence algebra is always a surjective image of the Hopf algebra of representative functions of a certain group built directly from the operad.

The Hopf algebra $\mathcal{Y}Sym$ has been fit into the construction of Moerdijk and Van der Laan, which builds Hopf algebras from operads [20],[27]. It seems very likely that this construction can be applied to operad bimodules in order to construct Hopf modules. We plan to explore this route both for its own interest and in order to get a clearer picture of the Hopf module of painted trees–since the multiplihedra do indeed form an operad bimodule over the associahedra. Once verified on this known example, our newly developed method for constructing Hopf modules will be applied to other operad module structures. These include the characterization of the cyclohedra as a module over the associahedra, as described by Markl [19]. The discovery of another module over the associahedra was published by the PI in 2008 [10]. This is the sequence of polytopes known collectively as the *composihedra*.

6. Graph Multiplihedra: generalizing painted trees.

6.1. **Graph multiplihedra and quotients.** The idea of painted trees is not hard to apply in general to tubings on graphs. We refer to the result as *marked tubings*. We completed an initial study of the resulting polytopes, dubbed *graph multiplihedra* $\mathcal{J}_G$, published as [7]. That paper also describes the quotients of the graph multiplihedra which generalize the composihedra, and the corresponding quotients which arise from forgetting the tube structure within the painted region. Examples for the case of the edgeless graph on three nodes are seen in the central lower four maps of Figure 12. Since then, a pair of new revelations has revealed the complete analogy to the classic multiplihedron structure.

6.1.1. *Generalizing the Saneblidze-Umble map.* Recall that Saneblidze and Umble described a cellular surjection β from the permutohedra to the multiplihedra in [25], and that this map was used to transfer the algebra structure of $\mathfrak{S}Sym$ to that of $\mathcal{M}Sym$ in [13] and [12].

Since the permutohedra are precisely the graph multiplihedra of the complete graphs, there is automatically a cellular surjection from $\mathfrak{S}_n$ to any $\mathcal{J}_G$ for graph G on n nodes. The map is described by deleting edges. At each deletion we create the induced marked tubing, and at the end of the process we have a marked tubing of the graph G. Figure 11 demonstrates this process.

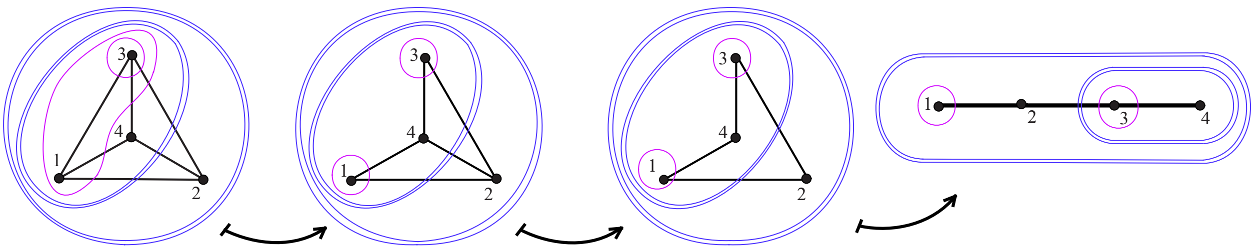


Figure 11. The surjection β factored. The corresponding cellular projections are of 4-dimensional polytopes.

Immediately this description of the Saneblidze-Umble projection suggests that there are a series of Hopf module algebras filtering the Hopf algebra of permutations, and which contain the Hopf module of painted trees. These have bases made up of the maximal marked tubings of the graphs in the restrictive and hereditary sequences described earlier.

6.1.2. *Special Factoring: edge deletion by tube marking.* The 3-dimensional case of the map β is shown factored in the top portion of Figure 2. There is an analogous picture for any graph. Thus for any of the generalized Saneblidze-Umble projections from $\mathcal{P}_n$ to $\mathcal{J}_G$ we can describe a factoring that is distinct from the one in Figure 11. In that figure we delete some edges and the projection equates two marked tubings which yield the same induced tubing after the deletion. With a certain set of edges in mind for deletion, we can factor the entire projection by first equating the tubings that originally differed only within thick tubes, and then equating the tubings that differed only in thin tubes. Or we could factor in the other order. For an example of the two options see the top four maps of Figure 12.

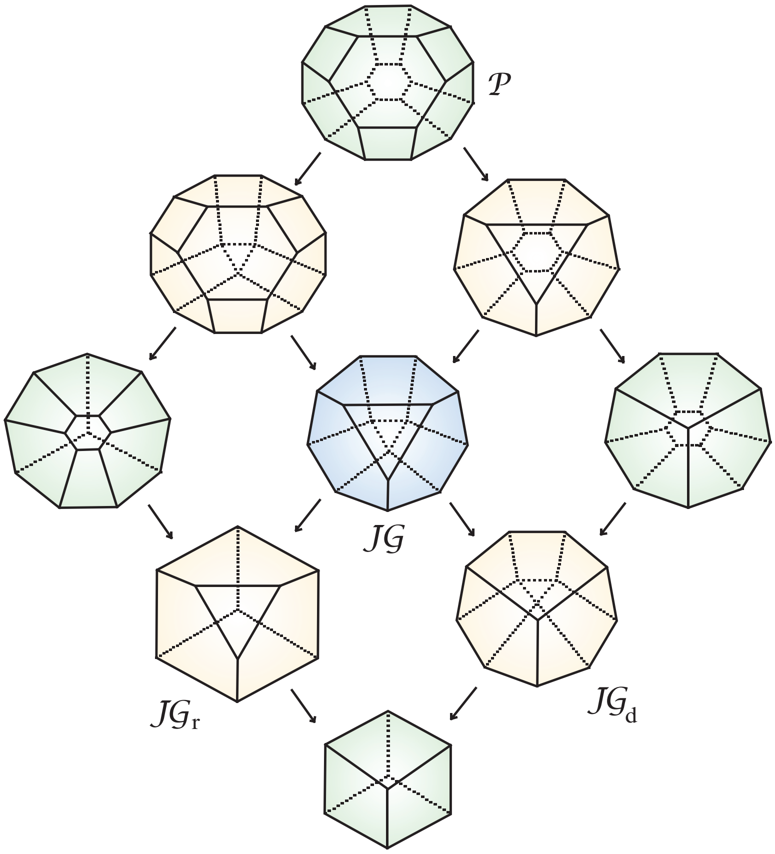


FIGURE 12. The commuting diamond of polytopes for G the edgeless graph on three nodes.

Of course the plan is to study the algebraic structure of these new polytopes.

Conjecture 6.1. *Given a hereditary (respectively restrictive) sequence of graphs, the faces of corresponding polytopes at any one of the nine locations of the commuting diagram exemplified in Figure 12 form the basis of a graded algebra (respectively graded coalgebra.) In addition the new polytopes all have lattice structures on their 1-skeleta which generalize the Tamari lattice.*

7. MULTIGRAPHS AND CW-COMPLEXES: NEW POLYTOPE SEQUENCES.

7.1. **Multigraph associahedra.** The number of redundant edges in a multigraph is the number of edges which would need to be removed to make it a simple graph. We dicovered a generalization of associahedra to multigraphs, simply by defining a tube to be any connected subgraph containing at least one of the edges between each of its pairs of nodes. In collaboration with Mike Carr and Satyan Devadoss, we have found that these polytopes are simple and have dimension $= n - 1 + |\{redundant\ edges\}|$, where n is the number of nodes. Figure 13 shows some examples in 3d.

7.1.1. *Discrete geometry.* It will be important to understand how these new polytopes relate to the familiar ones. For instance, I have proven that $\mathcal{K}_G$ for G the graph with two nodes and n edges is equivalent to $\mathcal{P}_n \times [0,1]$.

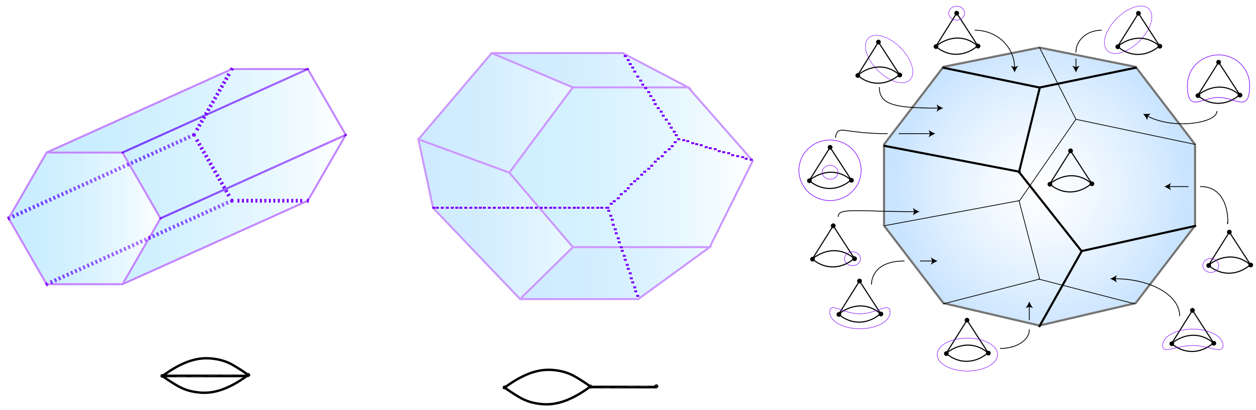


FIGURE 13. Several multigraph associahedra. The third is shown with labeled facets, and the reader may reconstruct the others in similar fashion.

We, especially collaborator Mike Carr, have discovered a method of truncation which yields the multigraph associahedron. The main purpose of this tool will be its use in gaining understanding of the recursive nature of the polytopes.

Question 7.1. *How can we characterize facets of a multigraph associahedron in terms of cartesian products of smaller multigraph associahedra?*

We believe that the answer to this is crucial to a full understanding of the algebraic nature of these polytopes. From observations of how the products in $\mathfrak{S}Sym, \mathcal{W}Sym, \mathcal{Y}Sym$ and ΔSym depend on the recursive stucture of the polytopes, the answer to this question should clarify which sequences of multigraphs support algebras and coalgebras. Another route to better understanding lies in finding how they fit into the picture of generalized permutohedra developed by Postnikov and Zelevinsky. Graph-associahedra are examples of Postnikov's *nestohedra*, and the collection of tubings on a graph exemplify a *nested set* [22]. In fact, more specifically they are *graphical nested sets* as defined by Zelevinsky [28].

Conjecture 7.2. *The face posets of multigraph associahedra are graphical nested sets, and the multigraph associahedra temselves are nested polytopes.*

This conjecture is plausible in light of Proposition 6.4 of [28]: Every two-dimensional face of the nested polytope is a d-gon for $d \in \{3, 4, 5, 6\}$.

7.1.2. *Hopf algebras.* There are several sequences of multigraph associahedra that we can immediately use to construct graded Hopf algebras. They are found simply by adding redundant edges to the path graphs and complete graphs, as shown in Figure 14. Recall that the path graph associahedron is the classic associahedron, the basis for $\mathcal{Y}Sym$. We have a proof that the multigraph associahedron for the graph with two nodes and n multiedges is a prism on the n^{th} permutohedron. Therefore we are compelled to ask:

Question 7.3. *What sort of combination of $\mathcal{Y}Sym$ and $\mathfrak{S}Sym$ is described by the Hopf algebras based on the multi-path graphs?*

An example of the multiplication of basis elements in a multipath Hopf algebra is in Figure 15.

7.1.3. *Graded algebras and coalgebras.* Again we can immediately describe new graded algebras and coalgebras by adding redundant edges to the graph sequences that support graph associahedra algebras and coalgebras. In addition, there are new coalgebras that may be described using new sequences of multigraphs such as the third sequence pictured in Figure 14.

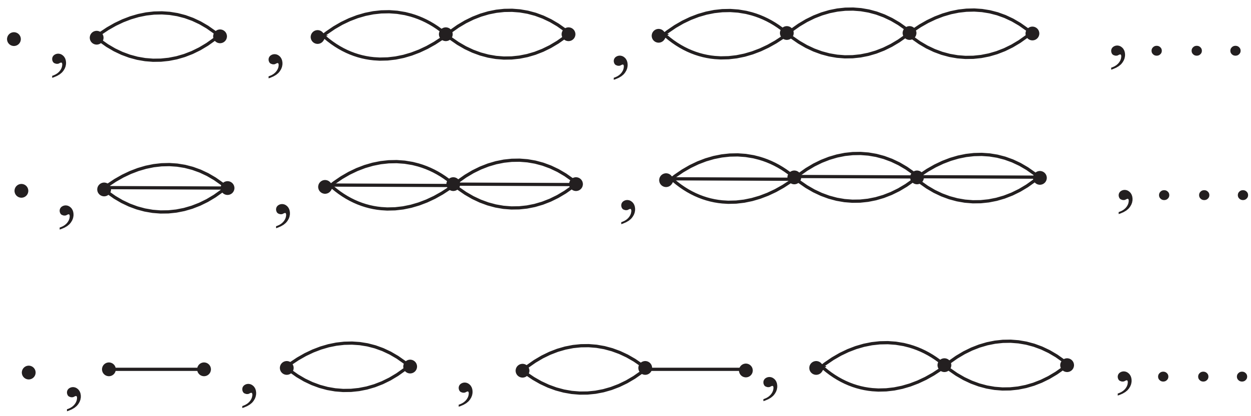


FIGURE 14. Some sequences of multigraphs which support new Hopf algebras (the first two) and a coalgebra.

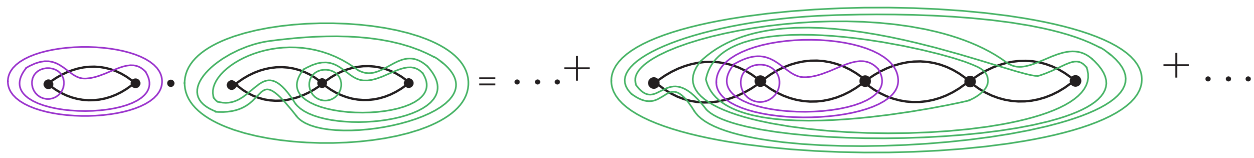


FIGURE 15. One term in the multiplication.

7.2. **Multigraphs with loops.** When the multigraph is allowed to have loops (which are counted as redundant edges) then the associahedron is no longer a convex polytope. Rather we find experimentally that the geometry is that of a polytopal cone. The dimension is as expected: the associahedron for a multigraph with n nodes and m redundant edges including loops is $n+m-1$. We have some preliminary results. For instance, the multigraph-associahedron of a single node graph with n loops is equivalent as a CW-complex to the quotient of the multigraph-associahedron of a two-node graph with n multiedges under identification of the two facets which correspond to its two trivial tubes.

7.3. **Cell-complex associahedra.** For each sphere S^k of any dimension contained in a CW-complex X, we count the number of higher dimensional cells attached by their boundary along that sphere and define the number of *redundant cells* to be one less than the total.

Definition 7.4. A tube T of X is a set of cells which span a connected subcomplex, such that for every set of dimension k-cells in T which form a copy of the k-sphere S^k, T includes at least one of the $(k+1)$-cells whose boundary is that copy of S^k.

Two compatible tubes are either nested or far apart. That is, T is compatible with T' if $T \subset T'$ or if $T \sqcup T'$ is not a tube. A tubing of X is a set of compatible tubes. Tubings are partially ordered by inclusion. For any CW-complex X, the corresponding CW-complex associahedron $\mathcal{K}_X$ is the polytopal cone whose face poset is isomorphic to the poset of tubings on X.

Conjecture 7.5. *In the experiments we have performed, for X a CW-complex with n 0-cells and m redundant cells, and all attaching maps homotopically nontrivial, the associahedron $\mathcal{K}_X$ is a convex polytope with dimension $n+m-1$. If there are p homotopically trivial attaching maps, then $\mathcal{K}_X$ will be a polytopal cone with dimension $n+m+p-1$.*

Example 7.6. Consider the *k-globular* cell complex G homeomorphic to S^k. It has two 0-cells (a copy of S^0), two 1-cells attached to that 0-sphere, two 2-cells attached to the resulting 1-sphere, and so on up to two k-cells. The polytope K_G is the $(k+1)$-dimensional cube. Note that the cell complex G' formed from G by the attachment of one more $(k+1)$-cell to make a $k+1$-dimensional disk also has $\mathcal{K}_{G'}$ the $(k+1)$-cube.

Example 7.7. Figure 16 shows a facet of a four dimensional CW-complex associahedron.

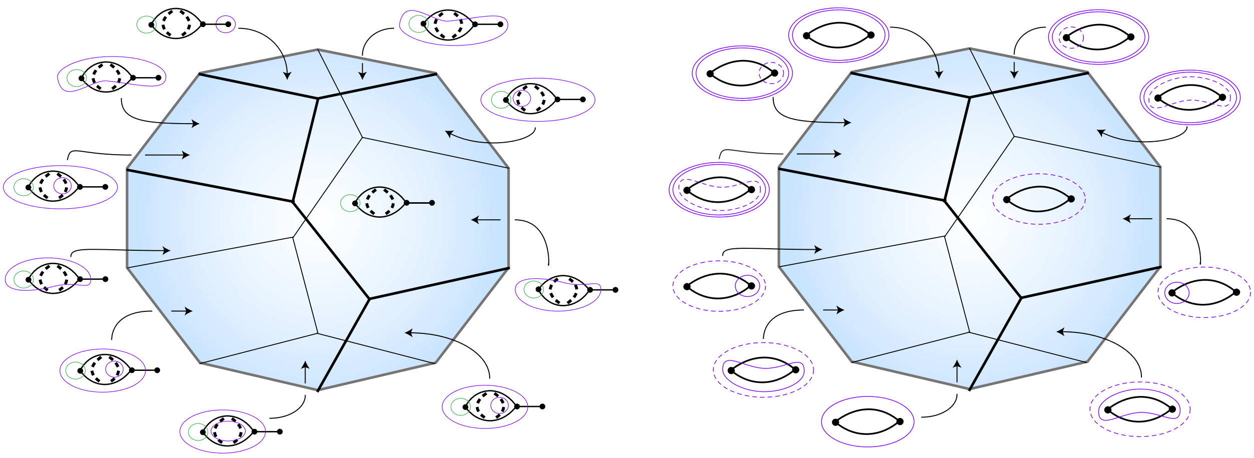


FIGURE 16. On the left is a single facet in the associahedron for the shown CW-complex. The CW-complex is constructed from three nodes, three edges, and then two disks (shown as dashed curves) attached to the copy of S^1. The facet corresponds to the singleton tube of the leftmost node. On the right, the same polytope appears as a multigraph multiplihedron.

The first example of an algebraic structure will come from CW-complexes made by gluing together a string of globular cell complexes of homogenous dimension to form globular paths.

Question 7.8. *Does the sequence of k-globular paths (whose n^{th} term has n 0-cells) give rise to a sequence of CW-complex associahedra whose vertices form the graded basis of a Hopf algebra combining the structures of $\mathcal{Q}Sym$ and $\mathcal{Y}Sym$?*

7.4. **Multigraph and CW-complex multiplihedra.** The *multigraph-multiplihedron* can be described simply by the poset of marked tubings on a multigraph. A marked tubing is a set of compatible marked tubes of the multigraph, with compatibility defined just as in [7]. An example is in Figure 16. We conclude with a conjecture and several questions germane to the existence of algebraic structures based on these objects.

Conjecture 7.9. *In the experiments we have performed, for G a multigraph with n nodes (loop free), the multiplihedron $\mathcal{J}_G$ is a convex polytope with dimension $n + |\{$ redundant edges $\}|$.*

Question 7.10. *Are the loop-free multigraph multiplihedra convex polytopes, and if so, can we describe their facets recursively as cartesian products?*

Question 7.11. *When we allow loops, are the multigraph multiplihedra polytopal cones?*

Question 7.12. *Are CW-complex multiplihedra realized as convex polytopes and polytopal cones? What is their facet structure?*

Question 7.13. *Are the range and domain quotients of multigraph and CW-complex multiplihedra convex polytopes?*

Question 7.14. *Do the 1-skeletons of the multigraph and CW-complex associahedra and multiplihedra form generalized Tamari lattices?*

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