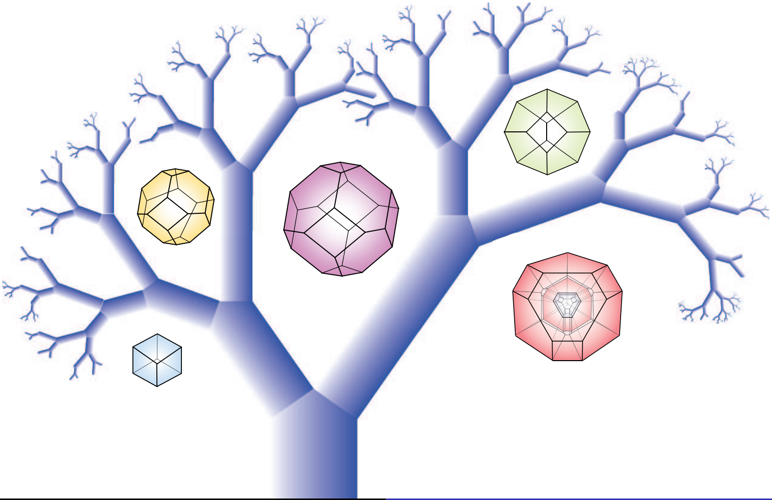
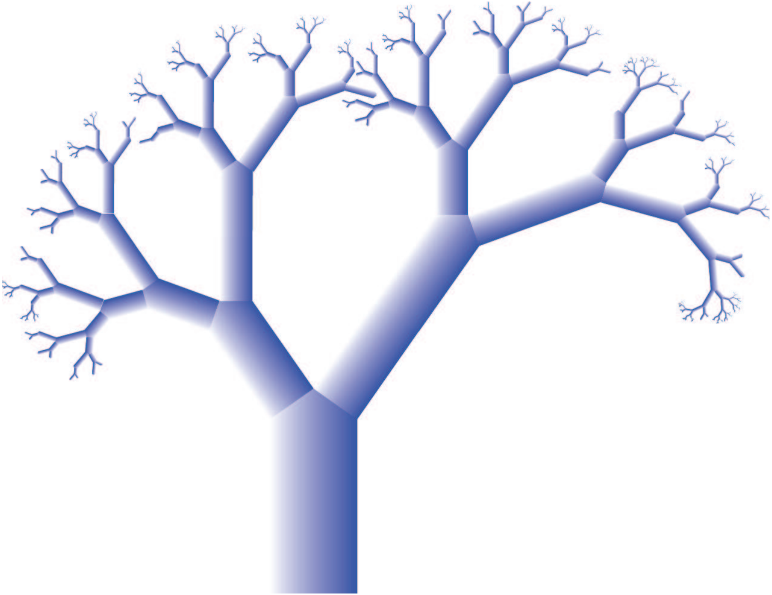
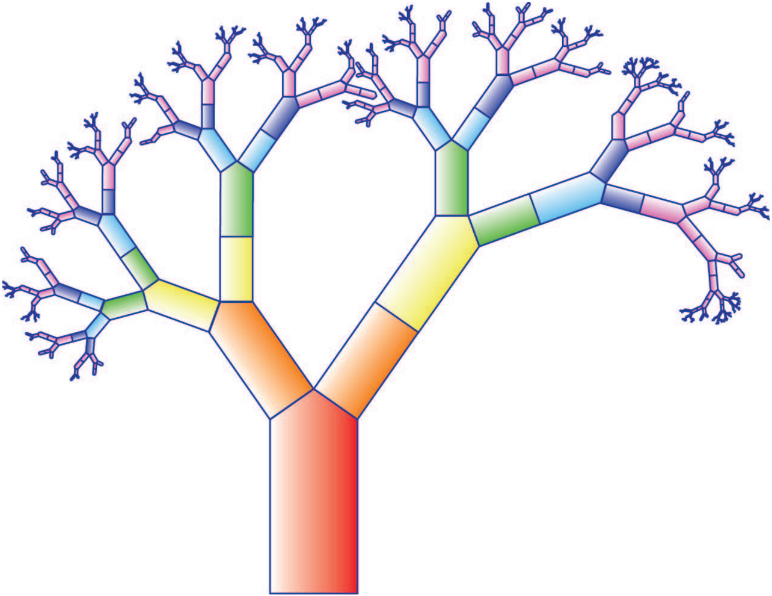
Trees and Polytopes   Stefan Forcey



Fibonacci tree $\mathcal{F}$.

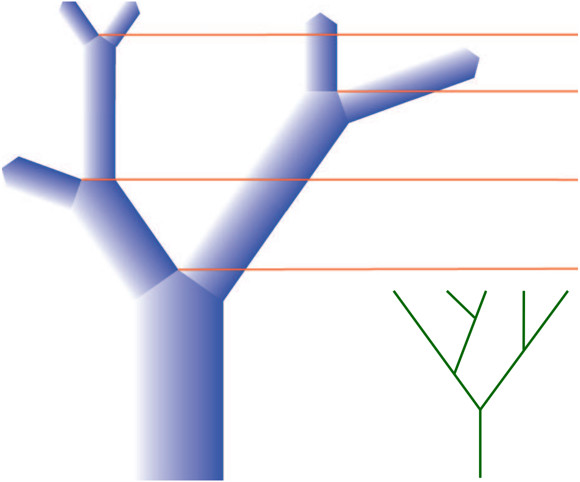


Fibonacci tree $\mathcal{F}$.

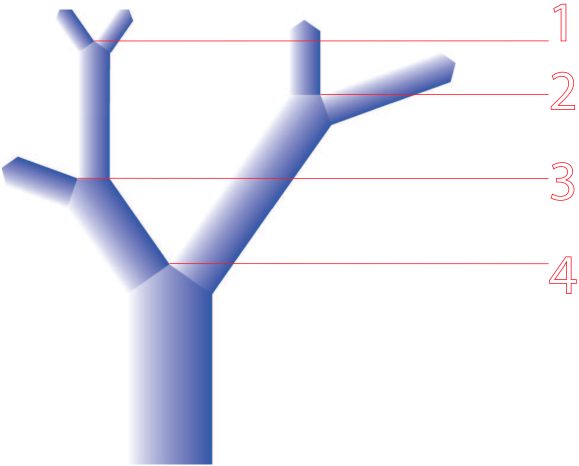


$\mathcal{F}$ is a *leveled* tree.

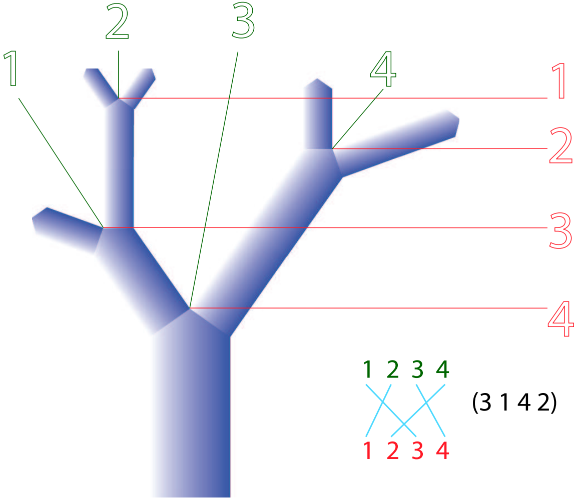
Leveled trees $\mathcal{S}$.



Leveled trees $\mathcal{S}$.

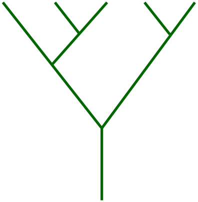


Leveled trees are permutations $\mathcal{S}_n$.

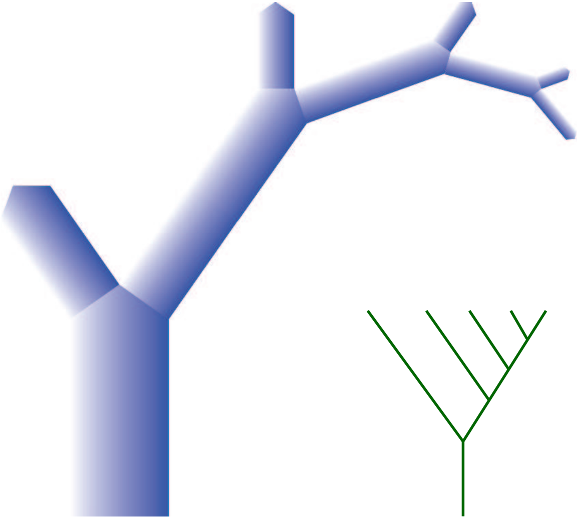


Q. What permutations are subtrees of $\mathcal{F}$?

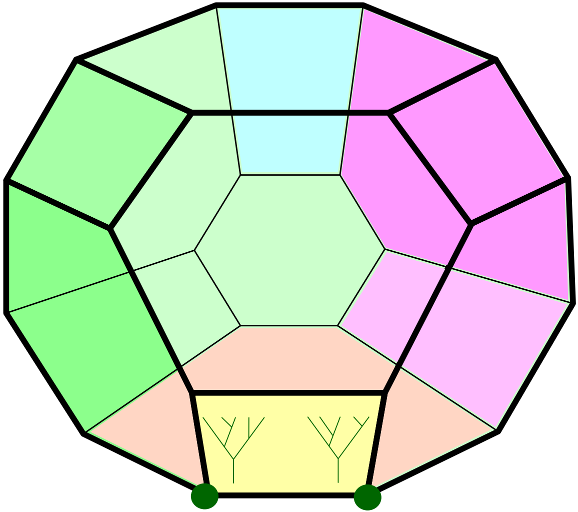
Binary trees $\mathcal{B}$.



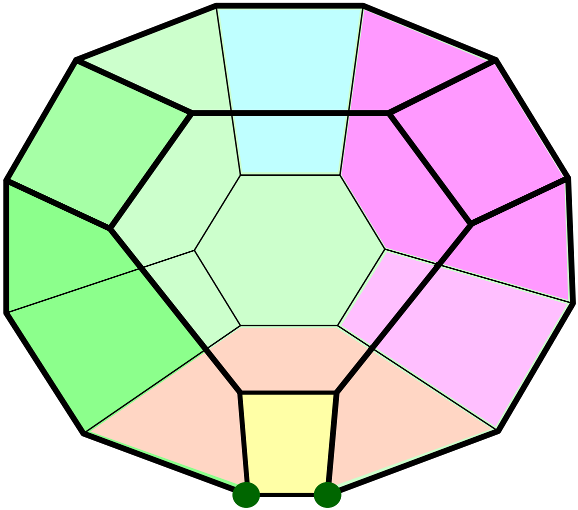
Combed binary trees $\mathcal{C}$.



Permutohedron.



Tonks cellular projection.



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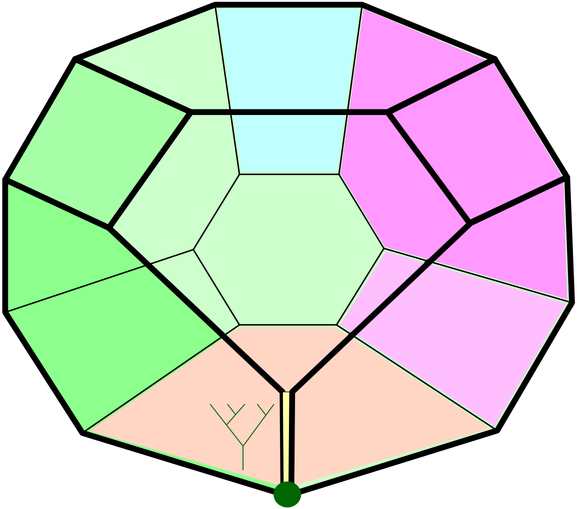

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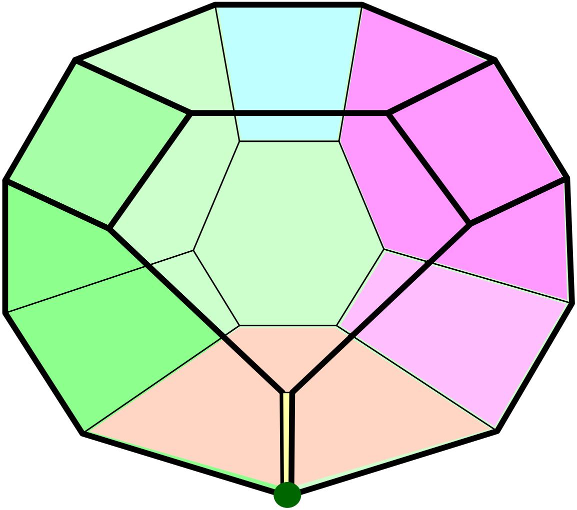

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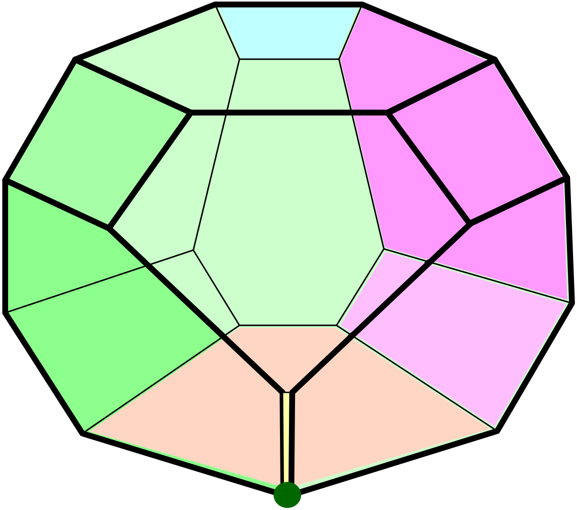
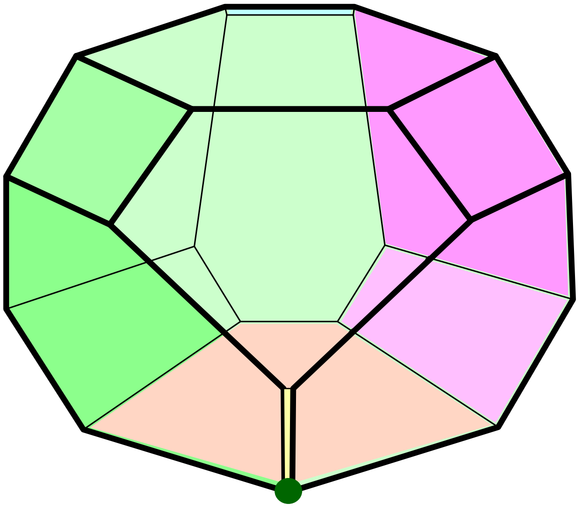


Tonks cellular projection.



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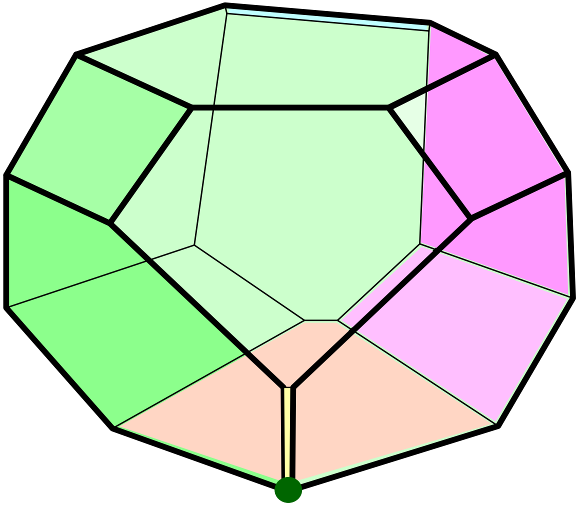

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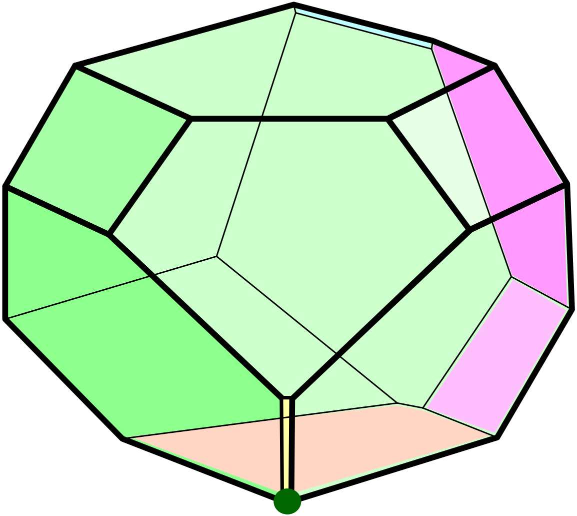
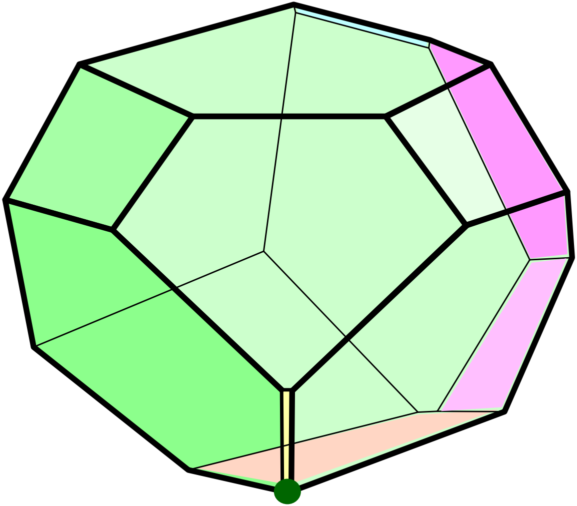


Tonks cellular projection.



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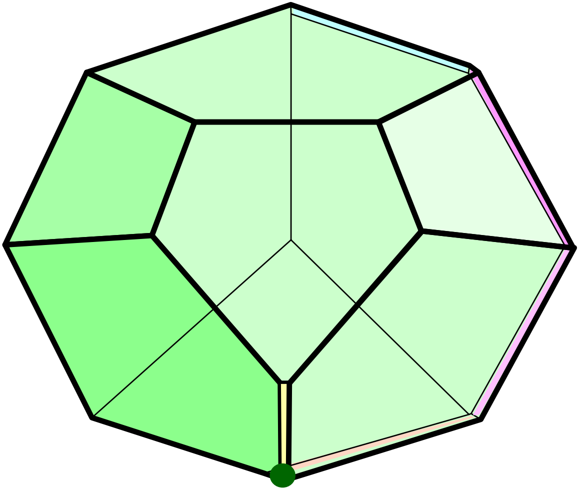
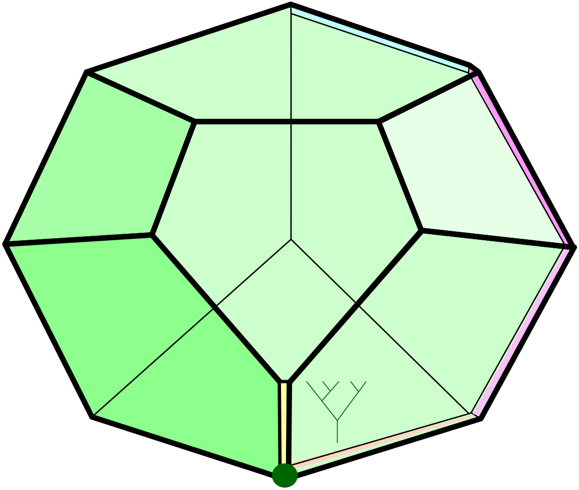
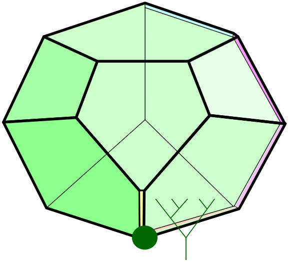


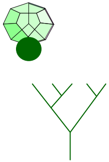
Tonks cellular projection.



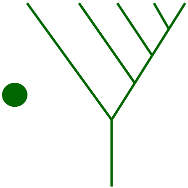
Projection...



Projection...



Projection...



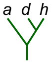
Species.

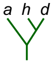
A *species* is a functor from Finite Sets to Finite Sets.

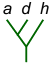
- *Example*: The species $\mathcal{L}$ of Lists takes a set to linear orders of that set.

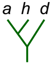
$$\mathcal{L}(\{a, d, h\}) = \{\ {\scriptstyle a<d<h,\ a<h<d,\ h<a<d,\ h<d<a,\ d<a<h,\ d<h<a}\ \}$$

- *Example*: The species $\mathcal{B}$ of binary trees takes a set to trees with labeled leaves.

$\mathcal{B}(\{a, d, h\}) = \{$ 

, 

, ... , 

, 

, ...$\}$

Species composition.

For a finite set U let $\mathsf{P}(U) =$ the set of partitions of U.

$$\mathsf{P}(U) = \{\{U_1, U_2, \ldots, U_n\} \mid U_1 \sqcup \cdots \sqcup U_n = U\}$$

We define the composition of two species:

$$(\mathcal{G} \circ \mathcal{H})(U) = \bigsqcup_{\pi \in \mathsf{P}(U)} \mathcal{G}(\pi) \times \prod_{U_i \in \pi} \mathcal{H}(U_i)$$

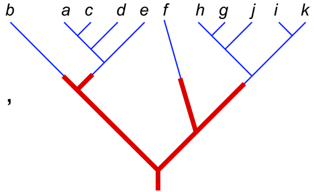
Familiar: also known as the cumulant formula, and the moment sequence of a random variable, and the domain for operad composition:

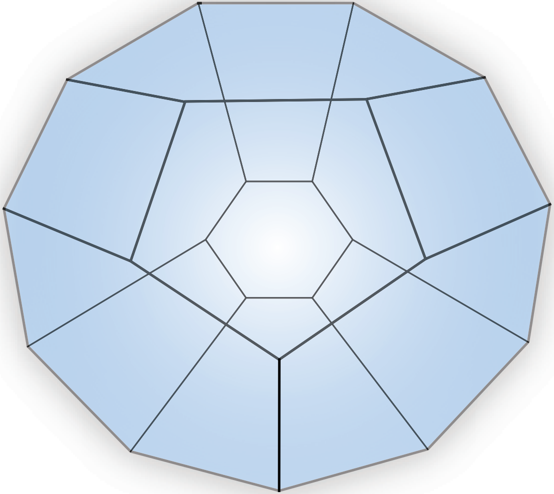
$$\gamma : \mathcal{F} \circ \mathcal{F} \to \mathcal{F}$$

Leveled tree of trees: indelible grafting.

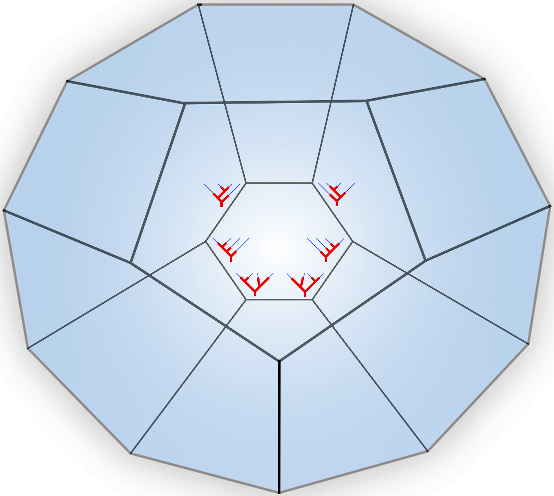
Example:

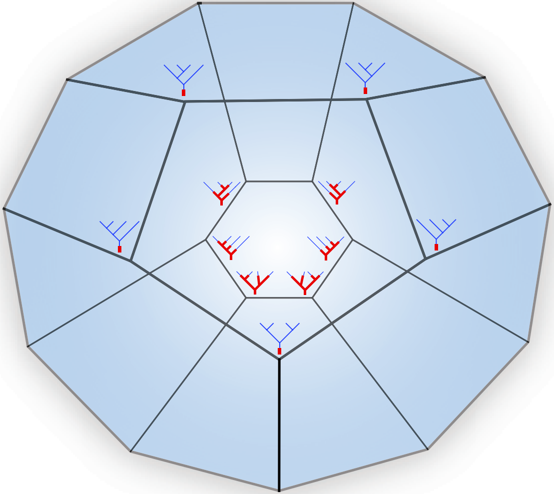
$$(\mathcal{S} \circ \mathcal{B})(\{a, b, c, d, e, f, g, h, i, j, k\}) =$$

$\{\ldots,$  $,\ldots\} = \{\ \frac{\text{trees}}{\text{leveled tree}}\ \}$

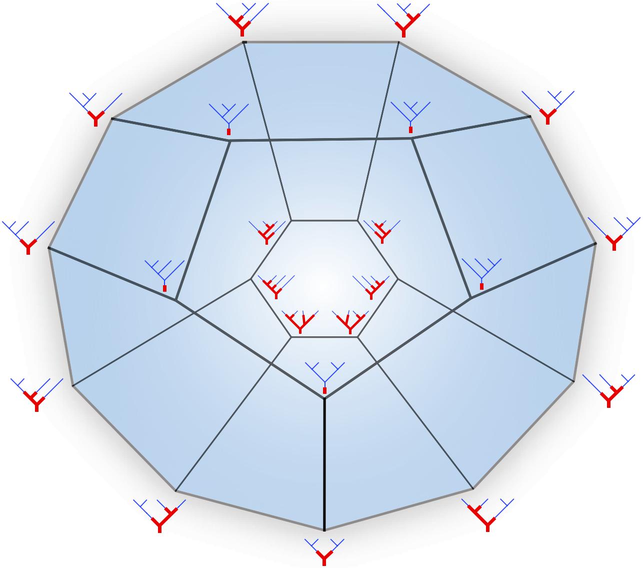


$\mathcal{S} \circ \mathcal{B}$





$\mathcal{S} \circ \mathcal{B}$



Composing species of tree.

$$
\begin{tikzcd}
 & & \mathcal{S} \circ \mathcal{S} \arrow[dl] \arrow[dr] & & \\
 & \mathcal{B} \circ \mathcal{S} \arrow[dl] \arrow[dr] & & \mathcal{S} \circ \mathcal{B} \arrow[dl] \arrow[dr] & \\
\mathcal{C} \circ \mathcal{S} \arrow[dr] & & \mathcal{B} \circ \mathcal{B} \arrow[dl] \arrow[dr] & & \mathcal{S} \circ \mathcal{C} \arrow[dl] \\
 & \mathcal{C} \circ \mathcal{B} \arrow[dr] & & \mathcal{B} \circ \mathcal{C} \arrow[dl] & \\
 & & \mathcal{C} \circ \mathcal{C} & &
\end{tikzcd}
$$

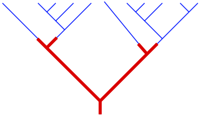
Composing species of tree.

$$
\begin{tikzcd}
 & & \mathcal{S} \circ \mathcal{S} \arrow[dl] \arrow[dr] & & \\
 & \mathcal{B} \circ \mathcal{S} \arrow[dl] \arrow[dr] & & \mathcal{S} \circ \mathcal{B} \arrow[dl] \arrow[dr] & \\
\mathcal{C} \circ \mathcal{S} \arrow[dr] & & \mathcal{B} \circ \mathcal{B} \arrow[dl] \arrow[dr] & & \mathcal{S} \circ \mathcal{C} \arrow[dl] \\
 & \mathcal{C} \circ \mathcal{B} \arrow[dr] & & \mathcal{B} \circ \mathcal{C} \arrow[dl] & \\
 & & \mathcal{C} \circ \mathcal{C} & &
\end{tikzcd}
$$

A small commuting diamond

$$\begin{tikzcd} & \mathcal{B} \circ \mathcal{B} \arrow[dl, two heads] \arrow[dr, two heads] & \\ \mathcal{C} \circ \mathcal{B} \arrow[dr, two heads] & & \mathcal{B} \circ \mathcal{C} \arrow[dl, two heads] \\ & \mathcal{C} \circ \mathcal{C} & \end{tikzcd}$$

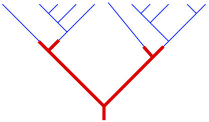
A small commuting diamond



$$\frac{\text{trees}}{\text{tree}}$$

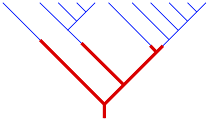
$$\begin{tikzcd}
 & \frac{\text{trees}}{\text{tree}} \arrow[dl, two heads] \arrow[dr, two heads] & \\
\mathcal{C} \circ \mathcal{B} \arrow[dr, two heads] & & \mathcal{B} \circ \mathcal{C} \arrow[dl, two heads] \\
 & \mathcal{C} \circ \mathcal{C} &
\end{tikzcd}$$

A small commuting diamond



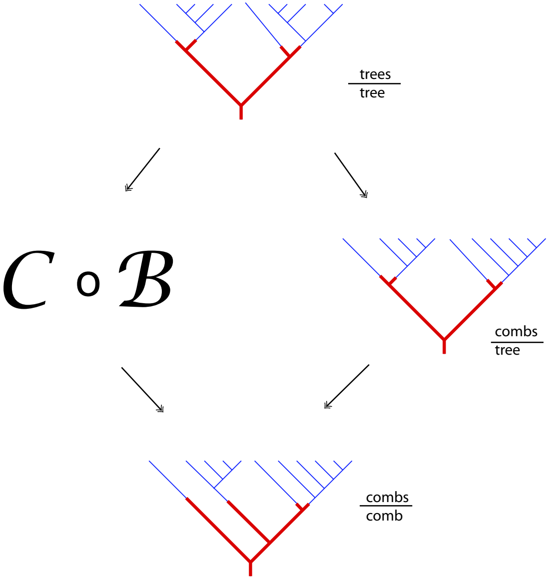
$\frac{\text{trees}}{\text{tree}}$

$\mathcal{C} \circ \mathcal{B}$ $\qquad\qquad$ $\mathcal{B} \circ \mathcal{C}$



$\frac{\text{combs}}{\text{comb}}$

A small commuting diamond



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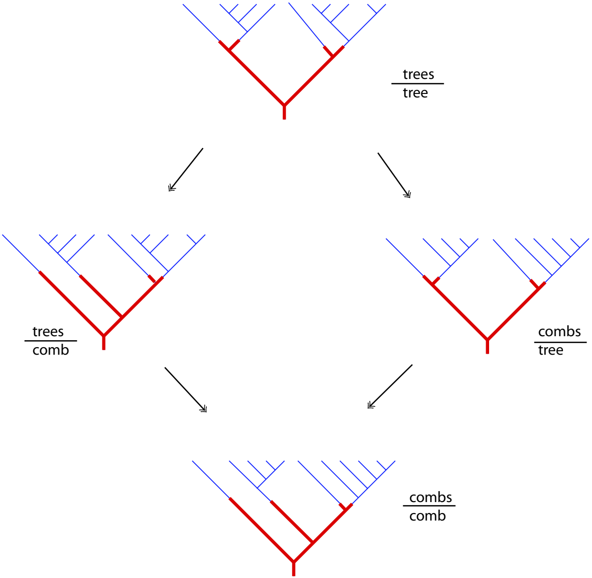
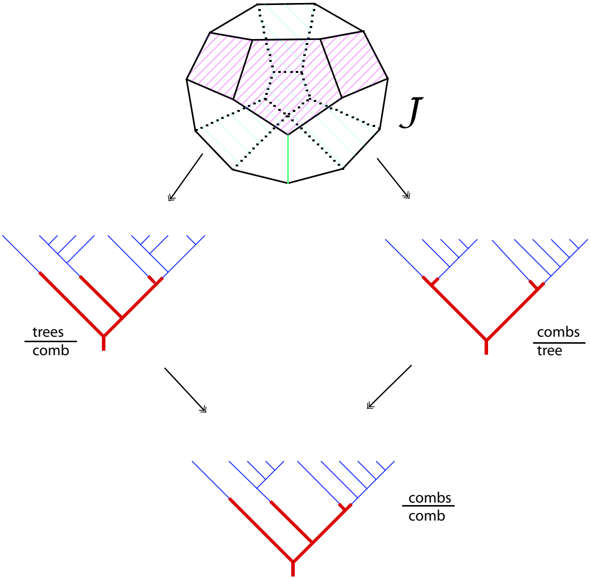
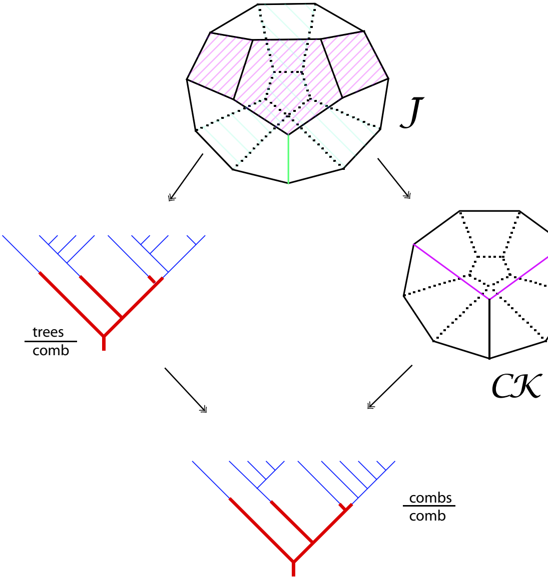


A small commuting diamond



A small commuting diamond



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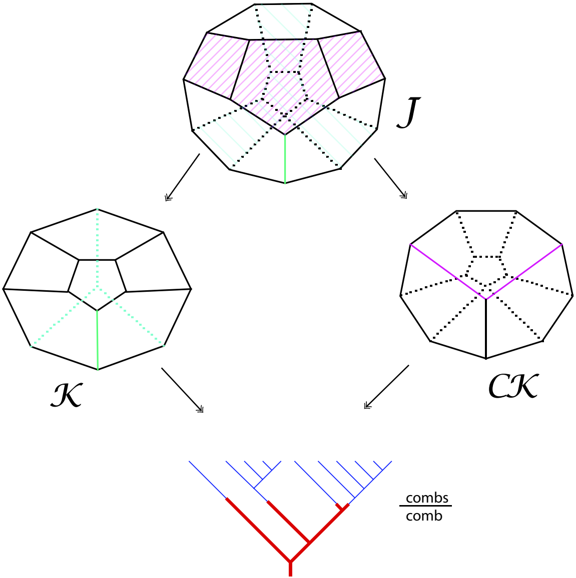

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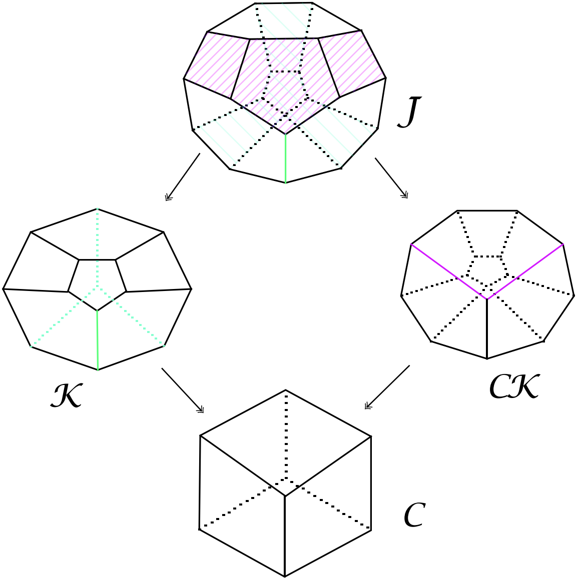
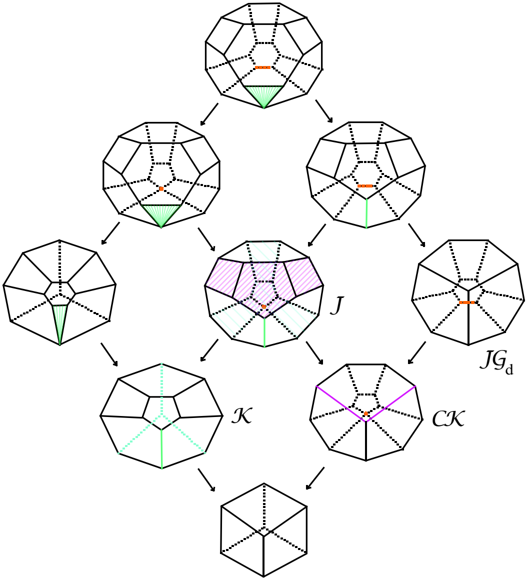


More polytopes.



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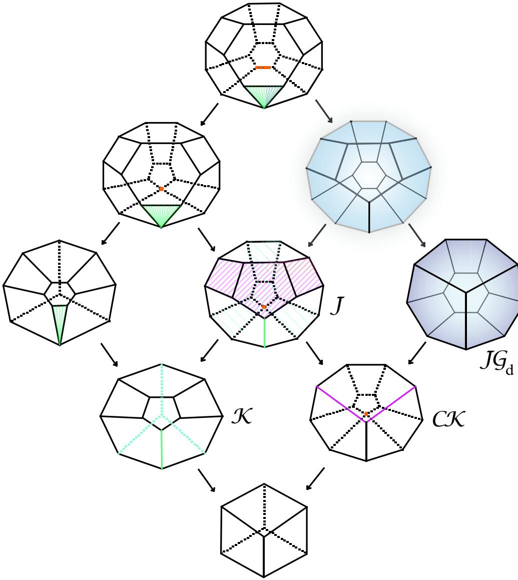
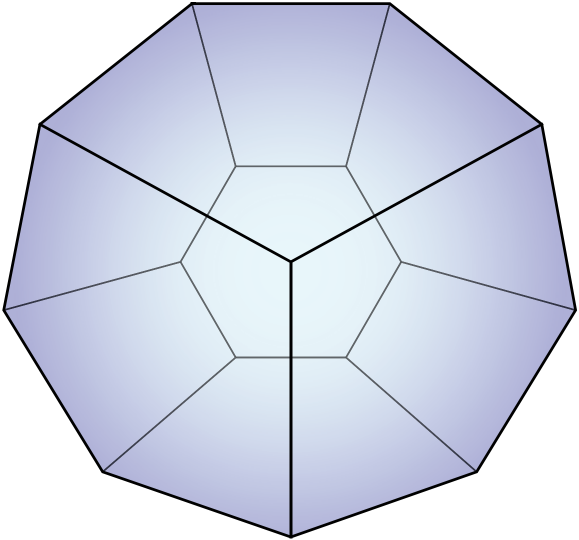
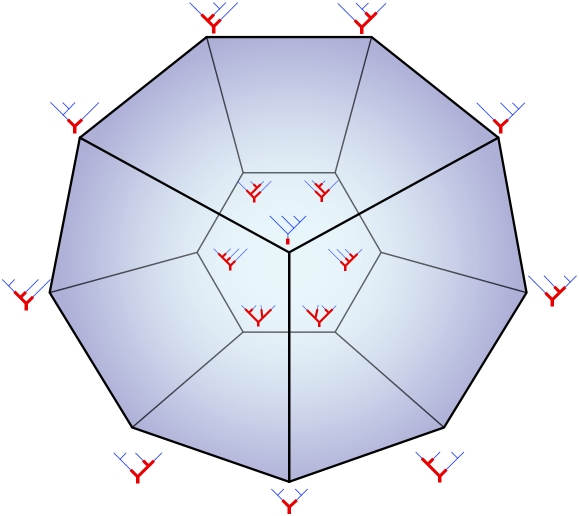


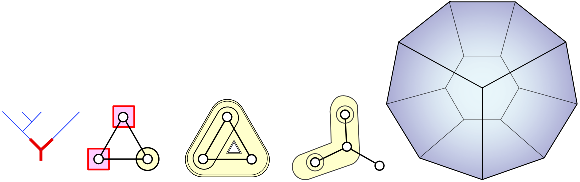
$\mathcal{S} \circ \mathcal{C}$



$\mathcal{S} \circ \mathcal{C}$



$\mathcal{S} \circ \mathcal{C}$



This polytope has been seen before!

Stellohedron = Complete-graph-cubeahedron

Number of vertices =

$$\sum_{k=0}^{n} \frac{n!}{k!}$$

Thanks!