Project description:
Finite type invariants and grope constraints on braids and string links.

1 Introduction and Examples

There are famous connections between the derived and lower central series of the fundamental group of a topological space and gropes continuously mapped into that space. The kth term in the derived series of the fundamental group of a space consists of the elements which are boundaries of the image of a map from a grope of height k to the space. The kth term in the lower central series of the fundamental group of a space consists of the elements which bound the image of a map from a grope of class k to the space. [56] In the latter case a candidate grope of shape 1/2 can always be supplied. Braid groups of n strands arise as the fundamental groups of configuration spaces of n points in the two dimensional disk under a quotient map identifying permutations of those points. Pure braid groups are the fundamental groups of the ordered configuration space of D^2. There are also hints of another geometrical characterization of the derived and lower central series of the braid groups. The subgroup of the pure braid group made up of all pure braids such that all (pairwise) linking numbers of the closure of the braid are zero is the second lower central subgroup of the pure braid group. In general, for the n-strand pure braid group, the Brunnian braids are a proper normal subgroup of the (n-1)–st lower central subgroup, with which they actually coincide up to link homotopy. [40],[41],[42]

The principal investigator has recently introduced a family of filtrations of the braid groups that potentially illuminates connections between these known facts. Consider a subset of braids on n strands for which there is also given a facet f of $\mathcal{K}(n)$, the Stasheff associahedron, which determines a parenthetical partition of the strands. The Stasheff associahedra are combinatorial polytopes whose vertices correspond to the complete partitions of a string of given length. Higher dimensional faces correspond to incomplete partitions which are compatible along shared lower dimensional faces.[108],[78] Furthermore, for each pair of matched parentheses in the partition of the string of length n that is associated to f, there is assigned a grope of type T represented by a binary tree. Grope annuli of type corresponding to a binary tree are CW complexes made up of a "cylinder" of any genus (surface with two circular boundary components) and with punctured tori iteratively attached along the meridians and longitudes of each doughnut hole in the cylinder according to the recipe encoded in the tree–right branch means longitude, left meridian. For initial purposes the gropes will be embedded in standard form (unknotted and not self-linked) in a 3-manifold. The constrained braids are those which allow the strands in each partition to be contained in an annular grope of the given type for that partition. Let the braid be embedded in $D^2 \times I$. Then in other words for a set of strands whose initial "input" points are enclosed by a pair of matched parentheses there is also a standardly embedded grope cylinder of the type associated to that pair of parentheses with one boundary surrounding the set of input points in $D^2 \times 0$ and the other the outputs in $D^2 \times 1$. The constraint is that no strands ever intersect the surface of the grope. The braid must also respect the partition associated to f by having the output points of the strands fall into the same partition, so that the braids for a given f and list of grope types form a subgroup of B_n. By respecting the partition it is meant that the permutation associated with the braid is such that it contains no factors which interchange positions from within and without any enclosing pair of parentheses of the partition.

Example 1 *Here is the $n = 4$ pentagon $\mathcal{K}(4)$ from the operad of associahedra.*

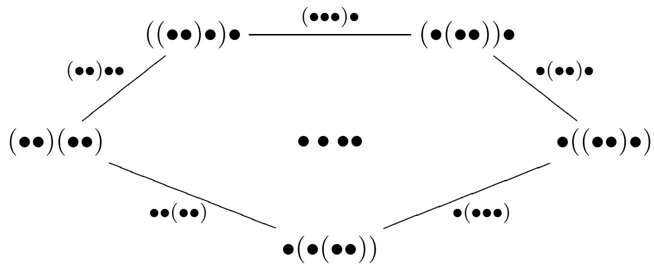

Figure 1 is an example of an element of B^T_{4f} where f is the side of the pentagon $\mathcal{K}(4)$ with corresponding partition shown and T is the grope type with corresponding binary tree shown.

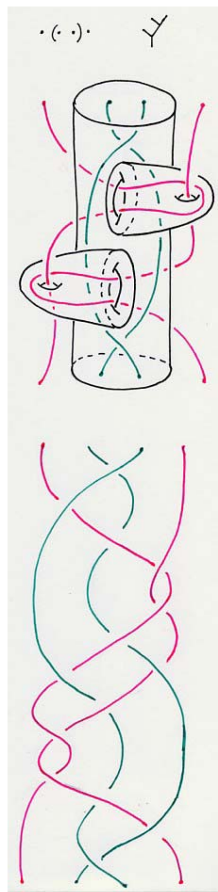

Figure 1: Typical element of a grope constrained subgroup of B_4.

Example 2 *Here is the associahedron $\mathcal{K}(3)$:*

$$(\bullet\bullet)\bullet \overset{\bullet\bullet\bullet}{\rule[0.5ex]{4em}{0.4pt}} \bullet(\bullet\bullet)$$

Figure 2 is an example of an element of B^T_{3f} where f is the vertex of the line segment $\mathcal{K}(3)$: with corresponding partition shown and T is the grope type with corresponding binary tree shown.

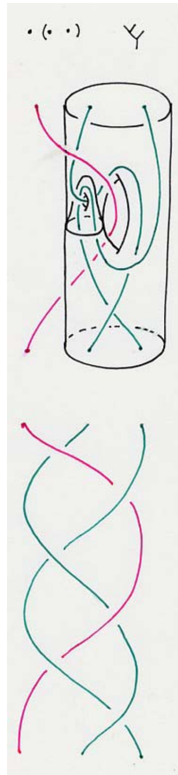

Figure 2: Typical element of a grope constrained subgroup of B_3.

Note that in Figure 2 the strands at first appear to be non-monotonic, but that this is just an artifact of trying to show the grope as well as the strands. It will be necessary to prove that whenever the strands do indeed form a braid up to isotopy that then the constraining gropes can be smoothly embedded around the monotonic strands. Note that if there are several sub-partitions of the same kind and not belonging respectively to separate sub-partitions, such as the two pairs in $\bullet(\bullet\bullet)(\bullet\bullet)$, then additional rules are needed to ensure the construction of a subgroup. Either it is necessary to insist that all sub-partitions end up where they began in the partition or if they are allowed to switch (as in Figure 8) it is required that unseparated sub-partitions of the same kind be assigned the same grope type. When the former restriction is applied the result is defined as semipure. The pure constrained braids are a special case of the semipure. The latter can be described as constrained braids whose constraining (grope) annuli when drawn with surface extrema extending from the top parentheses to matching bottom ones appear to be a pure braid diagram.

In order to attack open questions about finite type invariants, grope cobordism and concordance, it is of use to answer some fundamental questions about these grope constrained subgroups. This may include finding presentations for them, precisely describing the filtrations induced by the associahedra and the grope types and relating these subgroups to more familiar ones such as the Brunnian braids and commutator subgroups. It is also part of the project to investigate the link types that arise as closed elements of grope constrained subgroups. In this context Professor Forcey would like to find or recognize invariants that reflect the geometry of the constrained braids.

2 Presentations

Given a group extension G' of a group G by a group A, by which is meant a short exact sequence

$$1 \to A \to G' \to G \to 1$$

and presentations of G and A, there is a simple recipe for a presentation of G' as shown in [45]. In the case of a subgroup X_n of B_n there always is the exact sequence

$$1 \to PX_n \to X_n \to SX_n \to 1$$

where PX_n is the pure version of our subgroup X_n, i.e. intersection of X_n and P_n, and SX_n is the subgroup of permutations that can be achieved as the projections of braids in X_n onto the symmetric group. [57]

This reduces the problem of finding a presentation of a subgroup of constrained braids to finding presentations for the corresponding pure constrained braids and permutations. It seems that this problem will vary from relatively simple to quite difficult based primarily on the complexity of the grope types and secondarily on the number of strands. The hardest problems will correspond to facets of codimension $n/2$ in $\mathcal{K}(n)$. Some cases are straightforward. Figure 3 is Professor Forcey's initial conjecture about the generators of the pure braids of the first example above.

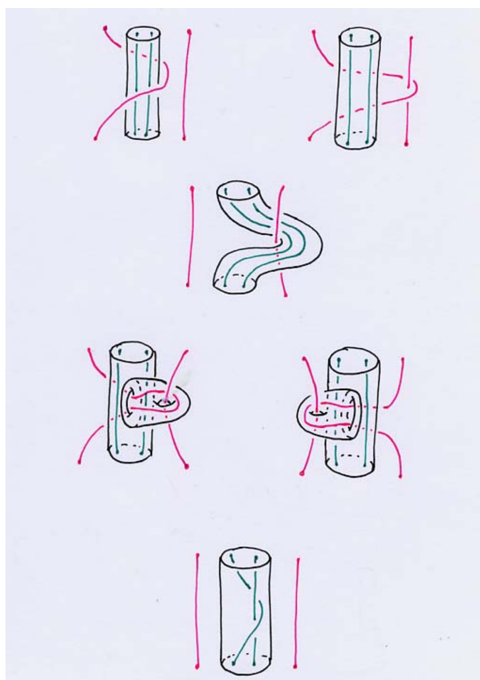

Figure 3: Generators of a pure grope constrained subgroup.

The subgroup of permutations will depend only on the facet of $\mathcal{K}(n)$. For the facet $f = \bullet(\bullet\bullet)\bullet$ as in example 1, the permutation subgroup SB^T_{4f} is presented by $< a, b | a^2 = b^2 = 1, ab = ba >$

For another example, in $K(5)$, consider the facet labled by $(\bullet\bullet)(\bullet\bullet\bullet)$. The permutation subgroup is presented by $< a, b_1, b_2 | b_1 b_2 b_1 = b_2 b_1 b_2, a^2 = (b_i)^2 = 1, ab_i = b_i a >$ where $i = 1, 2$. a is the interchange on the first two positions, b_1 switches the first two in the second group, and b_2 switches the last two.

There is much more work to be done here. The goal is to start with a facet of $\mathcal{K}(n)$ and a list of binary trees, add labels to strands and leaves, and then build a presentation out of those labels. Additional relevant material about presentations of related mapping class groups is in [39] and [58].

3 Filtrations

Given a facet of $\mathcal{K}(n)$ it seems clear that there are sequences of grope types that provide a filtration of the subgroup of B_n of braids for which the partition associated to the facet is preserved. This latter subgroup of braids can be described as being constrained by arbitrary genus cylinders. The smallest subgroup of the filtration is given by the case in which all the grope types are trivial, that is, the constraining surfaces are simply ordinary genus 0 cylinders. Figure 4 shows typical elements from an example of a filtration which includes the subgroup to which figure 1 belongs.

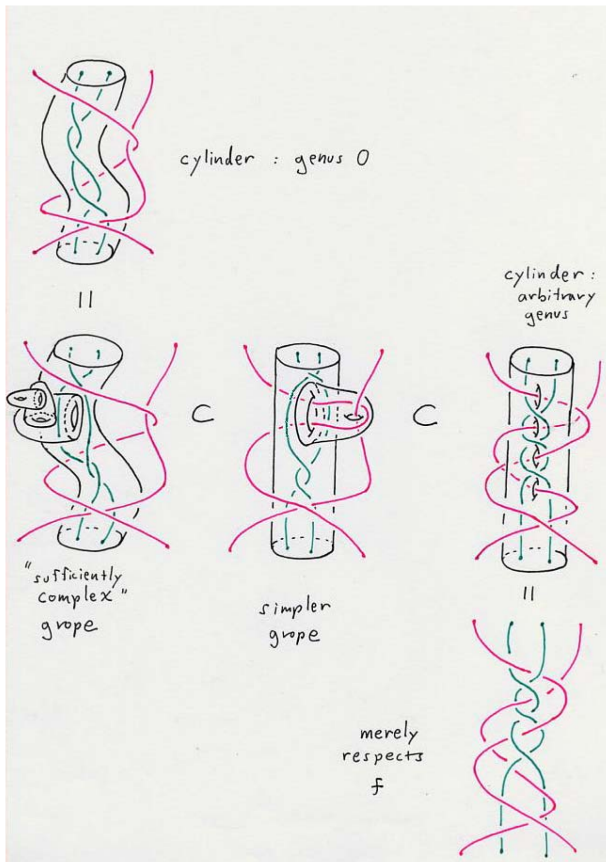

Figure 4: Filtration of braids which form the image of a certain subgroup of permutations.

It will be necessary to prove a precise theorem about the sufficient complexity of a grope to be equivalent to an annuli given the number of strands or constrained groups of strands within and without it. An initial conjecture is that any grope which includes surfaces attached to both longitudes and meridians is sufficiently complex for any strand configuration. This limits the available gropes to those with shapes of 1/2 and -1/2. For a constraining gropes of shapes +1/2, -1/2 and for respectively n exterior or interior strands, Professor Forcey conjectures that the class must be larger than n for the grope to be sufficiently complex.

Also, given a fixed grope type T, there is a filtration of B_n based on inclusion of faces in $\mathcal{K}(n)$. Figure 5 shows some typical elements from an example of this sort of filtration of B_4.

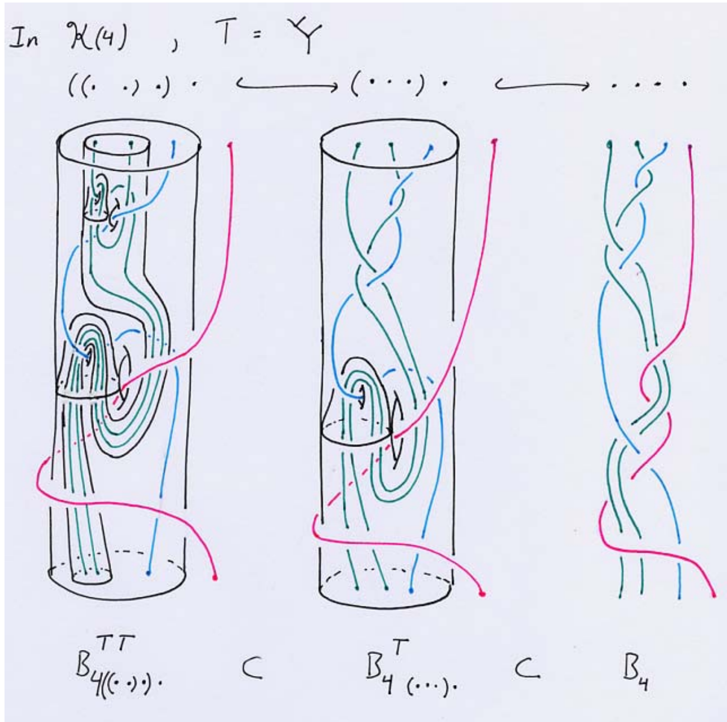

Figure 5: Filtration of braids that are constrained by a certain grope type.

There is potential for the filtrations above to be combined into a very comprehensive family of filtrations of B_n. This is in done by beginning with a complete parenthization of the strands, corresponding to a vertex of the associahedron. Then we loosely construct an $(n-2)$–dimensional array where $(n-2)$ is the number of pairs of parentheses. The positions in the array correspond to choices of grope type for each pair of parentheses, indexed by increasing complexity. Thus the first position in the array corresponds to the empty grope for each pair of parentheses, and gives B_n itself. Moving one step away from that origin in a given direction means introducing the trivial any-genus grope on that pair of parentheses which is equivalent to constraining the braid to respect that pair of parentheses by insisting that the associated permutation does not disturb the enclosed locations. Moving in any direction as long as the array indices increase or remain constant means we pass from a group to one of its subgroups. Eventually by increasing the array index for any pair of parentheses the grope type will be complex enough to be considered a simple annulus for the purposes of the constrained braid. The limit in all indices then is the completely constrained braid relative to the complete parenthization–all the gropes are simple annuli (class 1 gropes).

Notice that as in the following example (in Figure 6) that a braid is often found in two different nontrivial constrained forms, which are not directly related through one of the filtrations described above.

It may be important to know when to expect this sort of overlap, whether or not the intersection is a subgroup in turn and if so how to achieve its presentation. Nontrivial intersections occur both between subgroups that share a common vertex of the associahedra (and so occur on the same filtration array) and between subgroups in entirely different families.

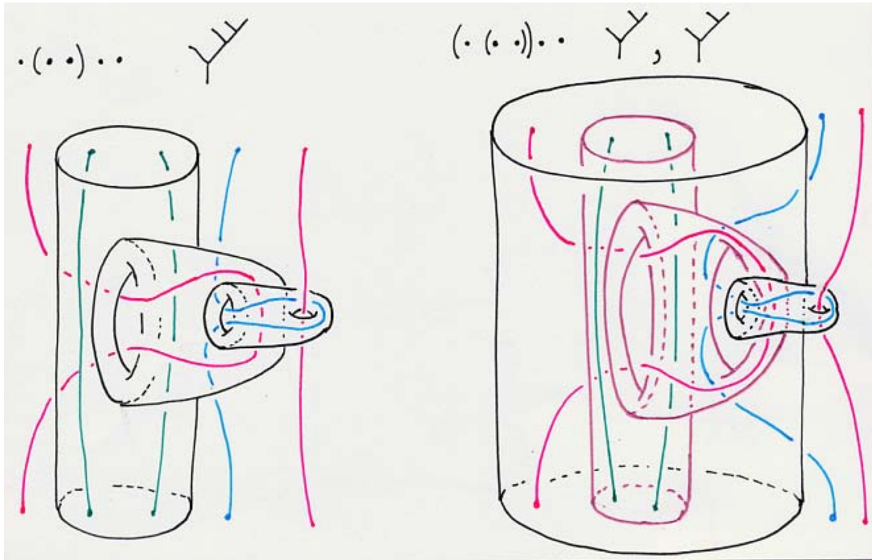

Figure 6:

4 Brunnian braids

Notice how Brunnian braids are often found as constrained pure braids. The basic example is the braid whose closure is the Borromean rings, shown in Figure 2 above.

Naively this phenomenon is due to the way that locally the constraining grope allows only "canceling interaction" between the strands inside it and out. That is, such pairs of separated strands have relative winding numbers of 0. More broadly, this hints at a subgroup relationship with the commutator subgroup and lower central series of subgroups of P_n. The constrained braid shown in Figure 1 lies in the second lower central subgroup of the pure braid group P_4. This latter braid also exemplifies a generalization of the Brunnian braids known as the k-trivial or k-decomposable braids, in which deletion of any k strands results in a trivial braid. Specifically, deletion of any two strands in the first example results in a trivial braid. Figure 7 shows another 2-trivial braid that is constrained by nested gropes. Again, since the actual relationship between Brunnian braids and the lower central series is only precise up to link homotopy of braids, it may be that the relationship between grope constrained subgroups and iterated commutators will follow suit. Besides [42] other relevant work has been described in [104], [46], [62] and [63].

5 Links

The closure of a constrained braid is a representation of a knot or link. Since the partition is preserved, the closure can be visualized as a link with closed gropes linked and knotted around and among its components. Within a constrained subgroup of B_n this picture suggests two variations of the equivalence on braids generated by link homotopy. First constrained link homotopy of braids would be the same as ordinary link homotopy, except with some limitation since strands are not allowed to intersect gropes. Thus the ordinary link homotopy classes would be subdivided. Secondly grope homotopy of braids would also allow changes of self crossing of closed grope components. Locally such a grope crossing change would appear as a crossing change of ordinary cylinders and their interior strands and nested cylinders all at once. Since a link component of a constrained link

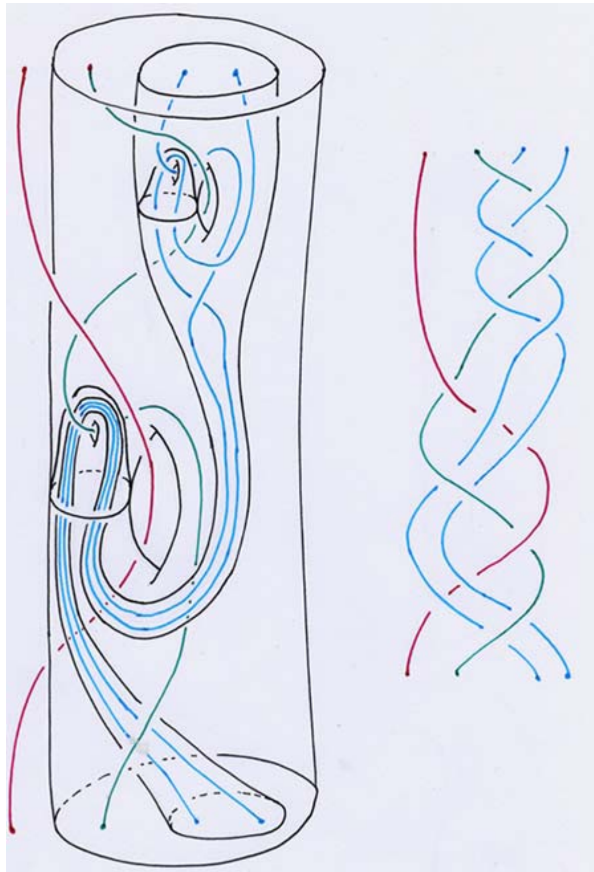

Figure 7: Example of a 2-trivial braid.

necessarily lies all within or all without each of the grope components, constrained link homotopy is a special case of grope homotopy.

To be a knot the partition of the strands must be "homogenous" by which is meant a partition into subsets of strands of equal cardinality. Otherwise, in order to respect the partition the braid is forced into having a closure with multiple components. Figure 8 is an example of a simple closed constrained braid that is isotopic to a recognizable knot–the connected sum of a figure eight and a trefoil. It is an intermediate goal to find link invariants that reflect the grope constraint. Upon inspecting a braid diagram and determining that it respects partitions (from a certain chain of included facets in the appropriate associahedron) it would be of value to be able to calculate invariant obstructions to types or classes of gropes that the braid might potentially be constrained by. This would be of interest especially if it afforded a new geometric understanding of existing well known invariants.

It is known that there is a close relationship between link homotopy, grope cobordism, and the number of components of a link. Two n-component links are link homotopic if and only if they cobound disjoint gropes of class n embedded in $S^3 \times [0,1]$. [56] It is of interest to determine how this relationship is reflected in the cases of grope and constrained link homotopy as described above. More relevant research on link homotopy and its invariants is to be found in [81],[82],[3],[5],[26], [31],[36],[42] and [41].

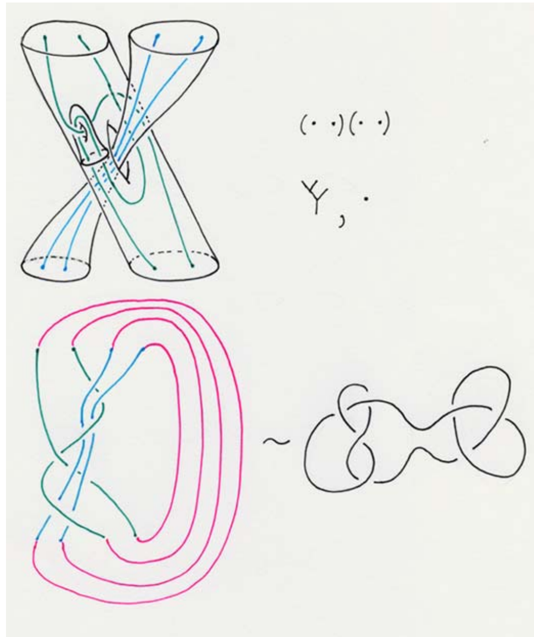

Figure 8: Closure of a grope constrained braid.

6 Goussarov–Habiro theory

There are several intimate connections between the lower central series of pure braid groups, homotopy theory, and Vassiliev invariants. Professor Forcey hopes to shed light on some outstanding related mysteries in part by investigating subgroups of braids constrained by gropes. For two knots K_1 and K_2 the following are equivalent:

(i) K_1 and K_2 have the same finite type invariants of Vassiliev degree $< n$.

(ii) K_1 and K_2 are cobordant by a capped grope of class n.

(iii) K_1 = the closure of a braid b and K_2 = the closure of pb, where p is in $LCS_n(P_k)$

Sources include:[105], [106], [96], [94], [95], [15] and [16]. Now since P_k is the fundamental group of $Config(D^2, k)$, p is thus the boundary of the image of a grope of class n mapped continuously into $Config(D^2, k)$. It is an open question bearing on the geometric meaning of the finite type invariants whether there is any direct relation between the various pairs of gropes of class n that occur above. For related material see [47]. In order to elucidate such a relationship the plan is to compare the filtrations given by grope constraints on the one hand and by the lower central series and derived series of the pure braids on the other. It may be necessary to pass to link homotopy in order to have a true comparison. It is expected that the constraining gropes will be comparable to the class n gropes in the equivalent statements above, enabling a search for canonical cobordisms and continuous maps respectively using and from the constraining gropes in question.

As shown in [107] points (i) and (iii) above also apply with finite type invariants replaced by delta finite type invariants and the pure braids replaced by the commutator subgroup of the pure braids. It is an open question whether there is a grope cobordism formulation of delta finite type invariants. Here again it seems likely that a precise comparison of braid commutators and constrained braids might uncover candidate gropes for cobordisms.

The equivalent three statements above lead to an equivalence relation on knots first formulated by Goussarov and Habiro, the latter of whom used tree claspers instead of gropes and the former of whom used certain crossing changes instead of pure braid composition. [33], [38], [95] These equivalence classes form abelian groups under connected sum of knots after identifying the connected sum of K and $(-K)$ with the unknot, but do not for links. It is an open conjecture that string links have the same behavior under composition with elements in the lower central series of pure braids and grope cobordism as do knots. They do form a nonabelian group under the equivalence relation of Goussarov. The string links are braids without the monotonicity requirement, and thus without existence of inverses. [36] Since monotonicity is not required of string links then gropes of higher complexity and symmetry contribute to the structure of grope constrained submonoids of string links than for braids of similar size. For instance, no grope of finite class can be sufficiently complex to be equivalent to a simple annuli constraint of a string link. The insistence on standard embedding of the gropes may also be relaxed for string links, allowing linked and knotted grope types. Two examples of constrained string links are pictured in figures 9 and 10. It seems wise

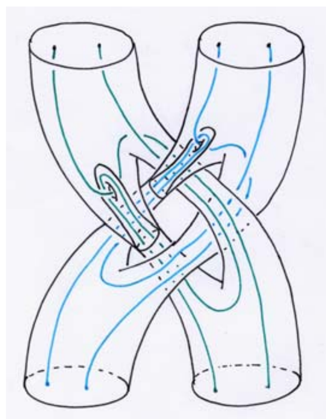

Figure 9: Grope constrained string link.

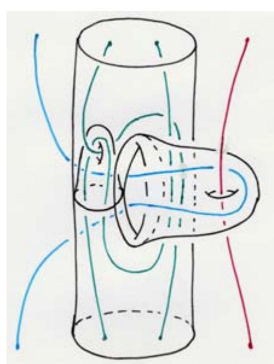

Figure 10: Symmetric grope constrained string link.

to look for a connection between symmetric grope cobordism, constrained string links, and knot equivalence modulo the derived series of the pure braids. First, recall that the kth term in the derived series of the fundamental group of a space consists of the elements which are boundaries of the image of a map from a grope of height k to the space. Symmetric gropes have positive integer height. Also note that the equivalence relation on knots induced by composition with elements of the derived series of the pure braids again gives rise to a group. It is unknown whether this equivalence relation is reflected by that given by symmetric grope cobordism, and to use grope constraints to investigate further it may be necessary to pass to string links.

Symmetric grope cobordism (in three dimensions) projects onto the filtration of the knot concordance group (in four dimensions) introduced in [13] and [12]. Compare with [50]. Knots that are related by composition with a pure braid are also related by a cobordism in $S^3 \times I$ introduced by the principal investigator. Precisely, if $K_1 = \overline{b}$ and $K_2 = \overline{pb}$ for $p \in P_k$ then there is a (genus 0) surface that is a cobordism from K_1 to $K_2 \bigcup \overline{p}$. The number of saddle points in this cobordism is k; each occurs as a component of $\overline{p}$ is connected summed with $\overline{b}$. Since $\overline{p}$ is a link of k unknotted components it would be nice to know the conditions on p such that it is concordant to the unlink, so that as a corollary K_1 is concordant to K_2. Link concordance is discussed in [26] and [30].

If concordance invariants can be constructed, perhaps as a limit of symmetric grope cobordism invariants, then the open questions of torsion in the concordance group could be attacked. There are also a series of conjectures of Conant and Teichner relating the knot groups to the filtration of the knot concordance group by grope cobordism in four dimensions. For instance, they conjecture that for a given collection of grope shapes, the two groups of equivalence classes of knots in 3 and 4 dimensions based on that collection are actually isomorphic. Further resources on knot concordance are [21], [8], [10], [18], [32], [48], [59], [65] and [75].

7 Categorical considerations

Categories equivalent to subcategories of the free braided monoidal category on one object are given such that objects are partitioned strings of the one object. The strands making up a morphism would be required to obey grope constraining. The braid need not however respect the partition exactly, since we do not require that all morphisms be composable. Morphisms then exist between objects that have the same string length and congruent partitioning, such as

$\bullet(\bullet\bullet)(\bullet\bullet) \to (\bullet\bullet)\bullet(\bullet\bullet)$

or $\bullet((\bullet\bullet\bullet)\bullet)(\bullet\bullet) \to (\bullet\bullet)(\bullet(\bullet\bullet\bullet))\bullet$.

Introduction of duals would allow us to speak of the subcategories of constrained tangles as well. As well as not enforcing monotonicity, tangles also allow a strand to begin and end at the same level, or to be a circle.

A "category of knots" point of view is especially well suited to the cobordism relation, since a surface can be seen as a morphism between disjoint collections of its boundaries. A 1+1 dimensional topological quantum field theory (TQFT) is determined in the case of a finite dimensional vector space range by its values for the disk, for the annulus, and for the "pair of pants," or trinion. It is a related goal of the principal investigator to formulate TQFT versions of finite type invariants by describing a field theory of grope cobordisms. This would involve in addition to the ordinary theory only showing how to assign an invariant to a grope annulus, since a grope trinion and a grope disk can both be cut into a grope annulus and an ordinary trinion or disk respectively. One possibility to investigate would be families of invariants generated by assigning to a grope annulus the group cohomologies of the various n-strand constrained subgroups of B_n or P_n associated to the grope type.

Since we are considering categories with 1 and 2 dimensional morphisms, we may indeed wish to bring to play the full power of higher dimensional category theory, using bicategories such as $\mathcal{V}$-Cat for $\mathcal{V}$ a monoidal category. Also of value may be the realization that the associahedra, the gropes and the braids all form operads indexed respectively by length of string to be partitioned, class and number of strands. It would be of much interest if the constrained subgroups arise as products of operads, or of operad algebras. Relevant resources include [49] and [78].

8 Homotopy groups of Spheres

The lower central series of braids also are important in sphere homotopy. For $n > 3$ the pure braid groups over S^2, with face maps deletion of strands and degeneracies doubling of strands, form a simplicial group with geometric completion of the homotopy type of S^2. There is a surjection from the Brunnian braids on n strands to the nth homotopy group of S^2. The nth homotopy group of S^3 is given by a quotient of the Brunnian braids on n strands over S^2. The Brunnian braids are cycles in the forementioned simplicial group. Since Brunnian braids often occur as grope constrained braids it is a good idea to look among the braids constrained by nested gropes for cycles and boundaries in regard to the simplicial structure. This is approximately a question posed by Fred Cohen in [14]. If quotients of grope constrained subgroups can be fit into a framework that realizes the homotopy groups of S^3 then it is tempting to generalize the process and look for relations between the quotients of grope constrained subgroups of braids over S^n and the homotopy groups of S^{n+1}.

9 Broader Impact: Research at an HBCU

The roles of researcher and instructor in mathematics are often seen to be at odds, one suffering when the other is focused upon. This is an unfortunate perception, since in actuality the quality and the motivational power of teaching at the university level is directly proportional to the instructor's involvement in leading edge research. The teacher/researcher is the link for the student between an esoteric world of developing science and the more familiar sphere of the classroom. Not only does the research activity of the professor keep his or her teaching relevant by forcing him to stay abreast of recent developments, but glimpses of the new results and unanswered questions he encounters energize his students with a larger view of their studies than afforded by the more mundane practice problems in their homework.

In particular the research into low dimensional topology discussed here will be performed largely at Tennessee State University, a historically black university with a large proportion of minority students. As well as helping to enrich the classroom instruction of the principal investigator this research project will further the participation of African Americans in mathematical research in several specific ways. The mathematics department at TSU requires a thesis from both its undergraduate senior mathematics majors and its masters degree candidates. Faculty advise students on their theses, and so active research projects such as the one being proposed are invaluable as sources of research topics for the degree candidates. The student benefits from having the experience of participating in new research and helping to develop new results and in having an adviser active in the field they are choosing to study. In addition, TSU is working towards the establishment of a Doctorate program in mathematics. Steps in that direction include the hiring of additional faculty actively involved in research, and the procurement of research grants to help continue that activity. As of this date there is only one PhD in mathematics offered by any of the historically black colleges and universities in the U.S., at Howard University. There is much to be gained from increasing this number in terms of broadening the participation of severely underrepresented ethnic groups in the mathematical community. The benefits to society of encouraging contributions to scientific research from all ethnic groups should be self evident. Whenever, for whatever historical or economic reasons, there is an underrepresented segment of society in an area of scientific endeavor, it means that inevitably valuable sources of talent and creativity are remaining untapped. The incalculable rewards for correcting this state of affairs are truly mutual.

On top of simply increasing the number of minority researchers in mathematics, the proposed project has as a partial goal the strengthening of ties between Tennessee State University and its counterparts in the region and wider academic community. The research proposed builds upon work done by T. Stanford, J. Hughes and J. Conant among others. These are some of the researchers in low dimensional topology that Professor Forcey has already begun to communicate with in regard to the work proposed. They are located respectively at New Mexico State University, Elizabethtown College in Pennsylvania and University of Tennessee in Knoxville. This last in particular houses the NSF Metacenter Regional Alliance for the advancement of computational science in historically black colleges and universities. While the project in question may not require large-scale computing facilities at least initially, it cannot but help encourage the networking of researchers within the Metacenter.

As results are finalized the principal investigator as well as potential student and faculty collaborators plan to disseminate the information through several venues. Articles will be prepared and submitted to appropriate scholarly journals, such as Knot theory and its Ramifications and Algebraic Topology. The latter, in which the principal investigator has published previously, is available online for free. Even before acceptance of journal articles, however, the material will be made available through preprint servers such as the arxiv and the Hopf server, as well as through conference and seminar presentations. Thus the answers to important mathematical questions discussed above will be easily available to those who have interest in the subject, its practical applications and potential further research.