2-DENDROIDAL SETS

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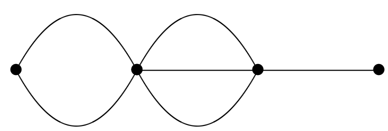
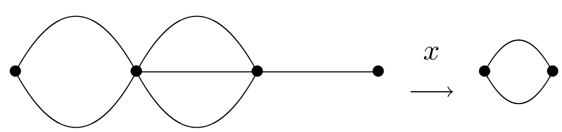
To describe the nerve of a 2-operad, we generalize the construction of the category Ω defined by Moerdijk and Weiss in http://www.arxiv.org/abs/math/0701293. Recall that the objects of Ω are trees and a morphism $t \to t'$ is an operad map from $\Omega(t)$ to $\Omega(t')$, that is, from the operad generated by the vertices of t to that generated by the vertices of t'.

[Globular pasting diagrams are pasting diagrams which correspond to trees with height. Note that the two sorts of trees just mentioned are not directly related. For one thing there are two sorts of tree composition around, one the ordinary grafting, and the other the special composition that reflects composition of pasting diagrams. We won't draw the trees with height unless they make definitions or proofs more efficient. Sources are Batanin [4] and Leinster [13]. For examples of pasting diagrams together with their trees see page 8 of [6] at http://www.arxiv.org/PS_cache/math/pdf/0301/0301221v1.pdf.]

Here is a pasting diagram in 2 dimensions.

$\pi =$ 

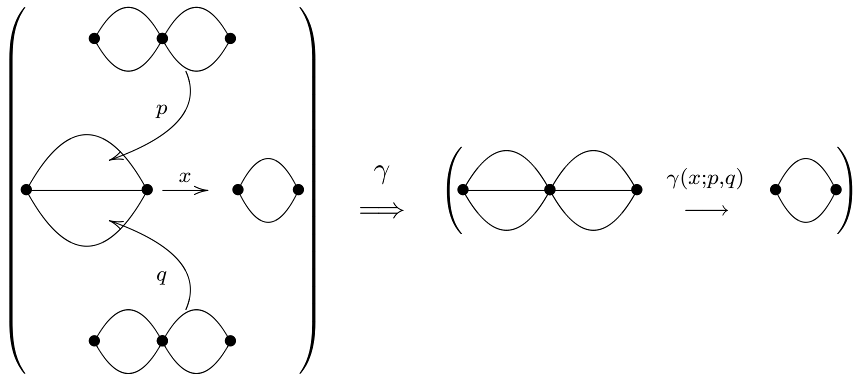
Now a 2-operad P of Sets has for each pasting diagram π a set $P(\pi)$. These will be non-coloured operads for now, since it seems extra complicated to start labeling all the cells of any dimension in a pasting diagram. An element x of $P(\pi)$ is drawn this way:



This drawing is meant to represent the idea that an element x of $P(\pi)$ can be seen as an operation taking data in the shape of the pasting diagram and composing to get a single 2-cell. The size of the single cell on the right is the upper bound size for the cells on the left. Composition in P is a map γ that takes an element of $P(\pi)$ together with an element from each of $P(\theta_i)$ (for pasting diagrams θ_i, one for each of the 2-cells in π, *which agree on their 1-dimensional source and target boundaries*) to an element of $P(\pi \circ (\theta_1, \theta_2, \dots))$ where the big pasting diagram is found by inserting the smaller ones into their respective 2-cells in π. Here is a picture of γ :

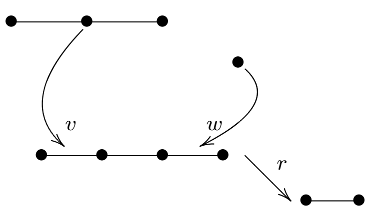
Key words and phrases. enriched categories, n-categories, iterated monoidal categories.

Thanks to XY-pic for the diagrams.

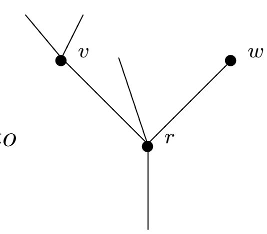


This drawing suggests using a picture like the domain of γ (left hand side above) to generate a 2-operad.

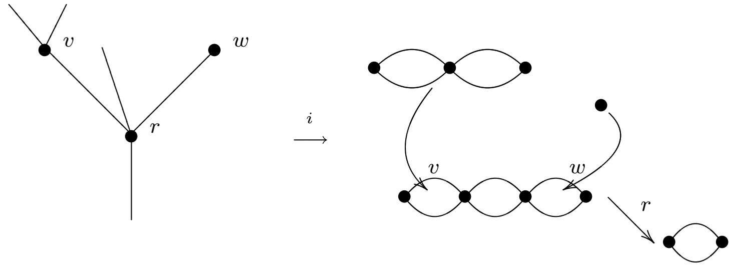
We'll call the relevant picture a *2-globular pasting tree* for now. In order to define a category Ω_2 of 2-globular pasting trees, for each 2-globular pasting tree t we need to construct a canonical 2-operad $\Omega_2(t)$. This construction will generalize the construction of $\Omega(t)$ when t is an ordinary tree, since an ordinary tree corresponds to a degenerate 2-globular pasting tree (1-dimensional) as in this example:



corresponds to

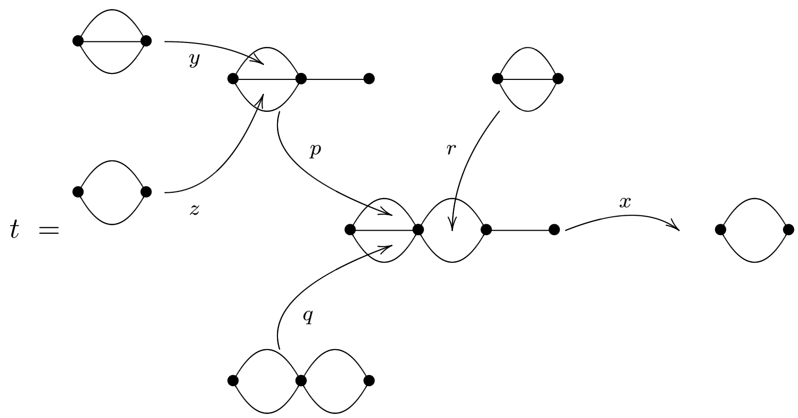


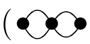
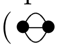
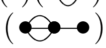
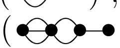
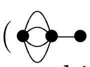
Therefore we can embed $\Omega \xrightarrow{i} \Omega_2$. The embedding is accomplished by taking a tree to the corresponding degenerate diagram, and then fattening it.



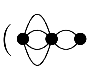
Now the definition of $\Omega_2(t)$ is analogous to the original definition of $\Omega(t)$. The labeled arrows of t generate the operations. Recall that we are considering non-colored operads.

Thus the operations are only indexed by pasting diagrams. Here is an example:



Now, for example, $q \in \Omega_2(t)($$)$;
$y, r \in \Omega_2(t)($$)$;
$p \in \Omega_2(t)($$)$;
$x \in \Omega_2(t)($$)$;
and $\gamma(p; y, z) \in \Omega_2(t)($$)$.

Now there is a question which I am not sure of the implications of. In this example, is it true that

$$\gamma(r; p, q) \in \Omega_2(t)($$$$)?$$

This would become clearer, perhaps, if we were in the realm of coloured operads or multicategories. Here we have a single color, and so here it seems that the answer is yes–but I rather think not, since the generated operad should respect the structure of the pasting tree.

Now the definitions proceed. Ω_2 is the category which has objects 2-globular pasting trees $t, t', \ldots$ and morphisms 2-operad morphisms $\Omega_2(t) \to \Omega_2(t')$ as defined in [4]. A 2-dendroidal set is a functor $\Omega_2^{op} \to \mathbf{Set}$. (Perhaps the image should be globular sets: there is an unspoken subplot of boundaries of pasting diagrams that we've put off discussing.) A map between 2-dendroidal sets is a natural transformation. The category of 2-dendroidal sets is denoted $d_2\mathbf{Set}$.

The restriction of 2-dendroidal sets to dendroidal sets is denoted $\mathbf{i}^* : d_2\mathbf{Set} \to d\mathbf{Set}$, and it takes a 2-dendroidal set to the sub-collection of sets which are indexed by the *2-simple* pasting trees–those with each node just a string of 2-cells composed along 0-cells. $\mathbf{i}_! : d\mathbf{Set} \to d_2\mathbf{Set}$ is given by:

$$\mathbf{i}_!(X)_t = \begin{cases} X_t, & t \text{ is 2-simple} \\ \emptyset, & \text{otherwise} \end{cases}$$

The dendroidal nerve of a 2-operad P is given by a functor N_2d from the category of 2-operads $\mathbf{Oper}_2$ to $d_2\mathbf{Set}$, where $N_2d(P)(t) = \mathbf{Hom}_{\mathbf{Oper}_2}(\Omega_2(t), P)$.

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