Cofree compositions of coalgebras: Trees, polytopes and indelible grafting.

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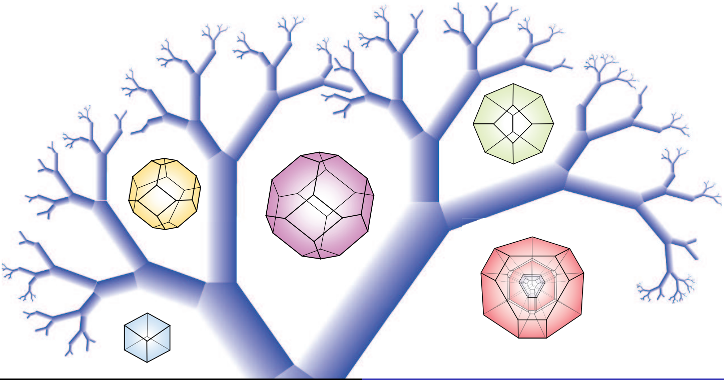
Frank Sottile, Texas A&M U.



Our slogan.

"Niceness is hereditary in species."

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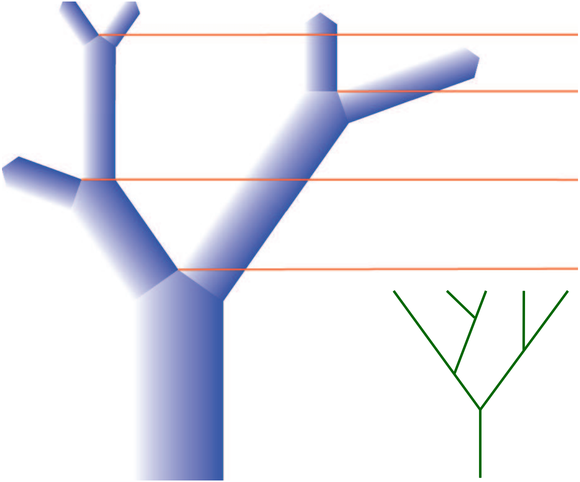
For this talk, nice properties of species of coalgebras will be:

1. Cofree-ness,
2. Hopf-ness,
3. Polytopal-ness.

But first, our cast of characters:

Ordered trees $\mathfrak{S}$.



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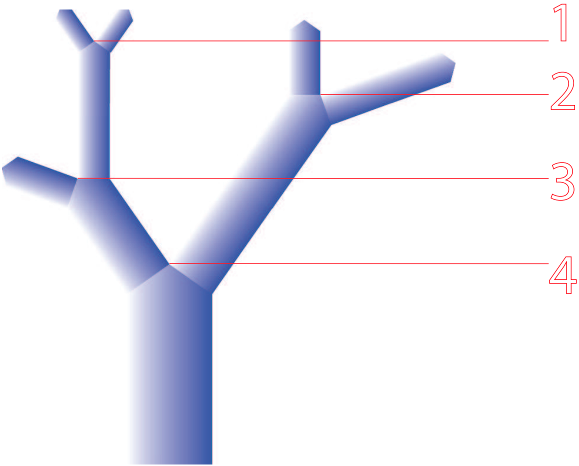
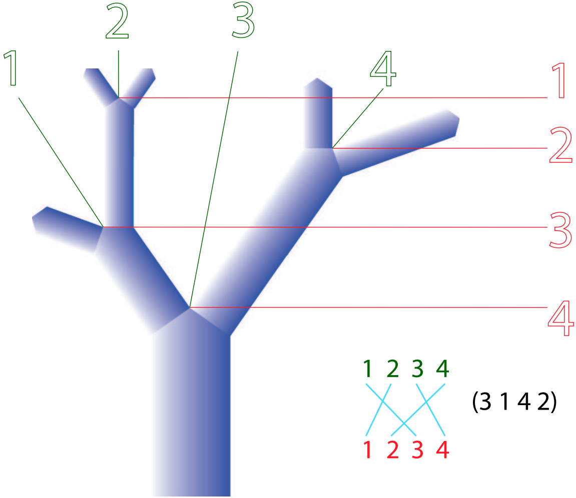
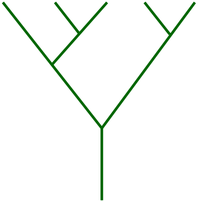


Ordered trees are permutations $\mathfrak{S}_n$.

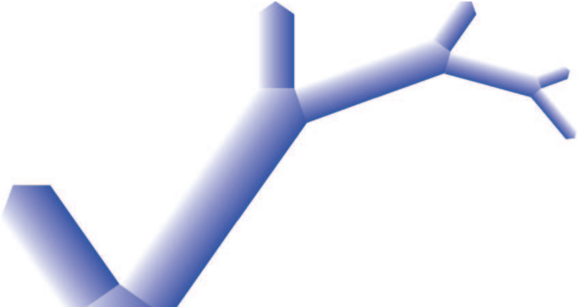


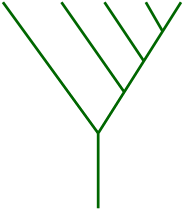
(3 1 4 2)

Binary trees $\mathcal{Y}$.



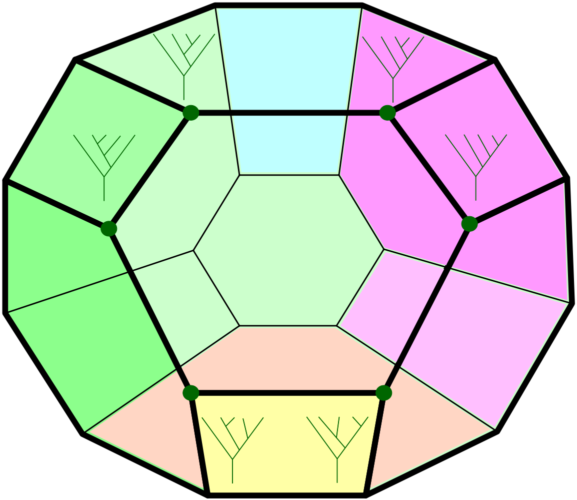
Combed binary trees $\mathfrak{C}$.



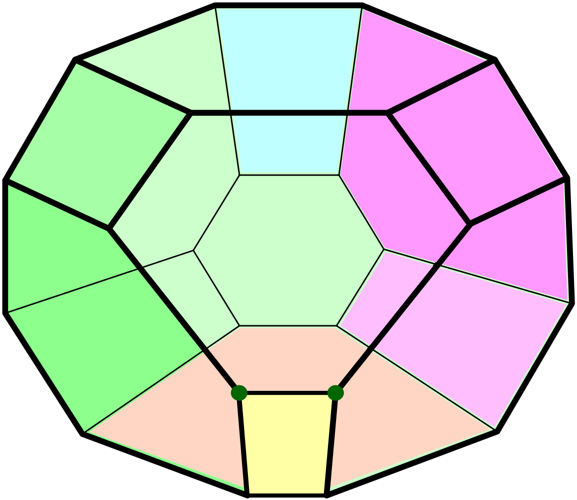


Our cast as Polytopes.

$\mathfrak{S}$: Permutohedron.



Tonks cellular projection.



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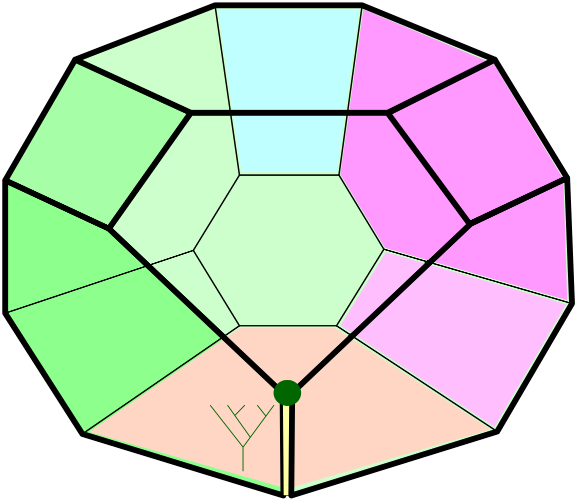

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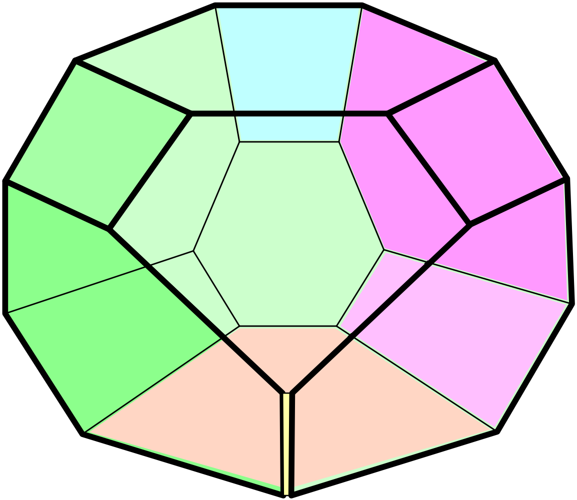

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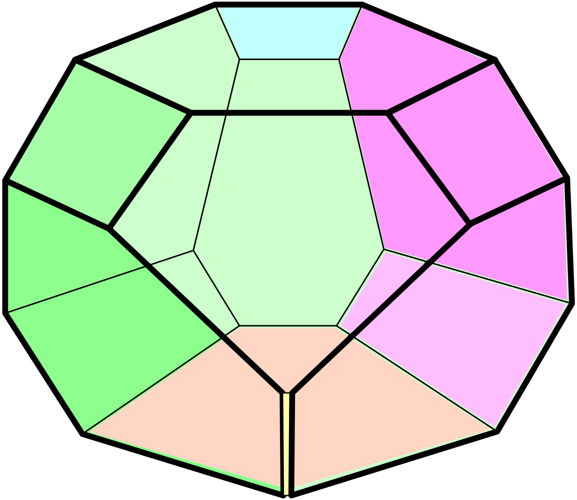

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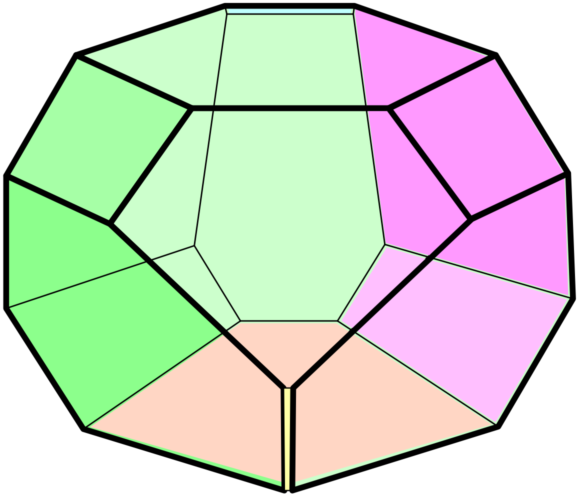

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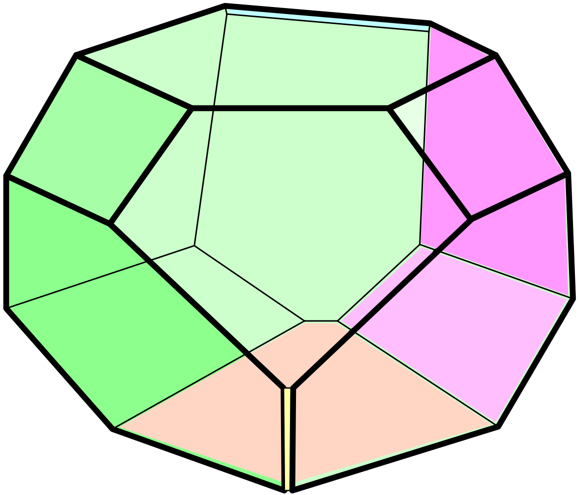

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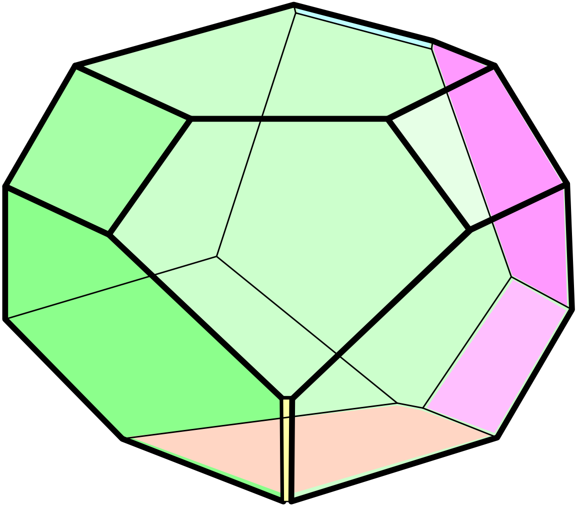

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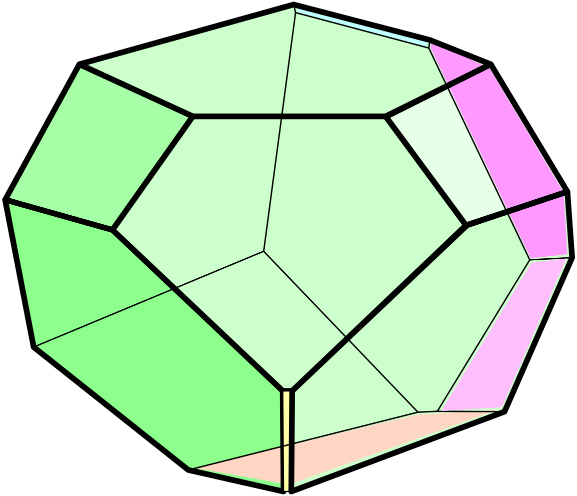
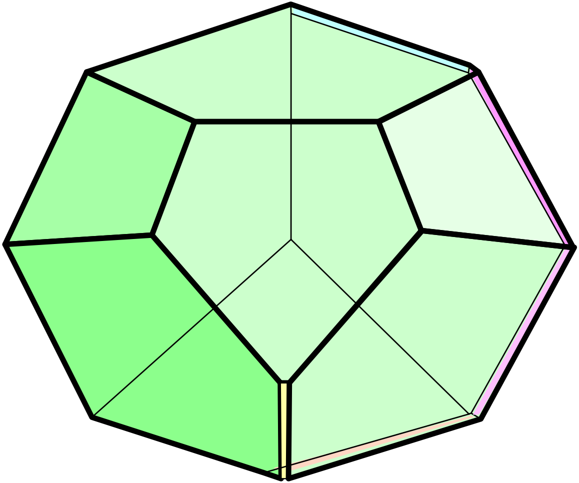


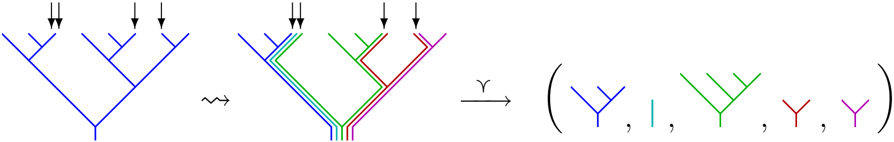
$\mathcal{Y}$: Associahedron



Our cast as graded Hopf algebras.

A Hopf algebra of binary trees.

Two operations on trees: splitting



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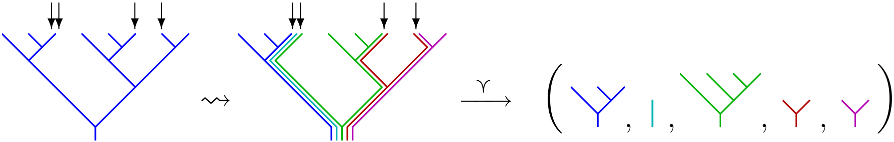
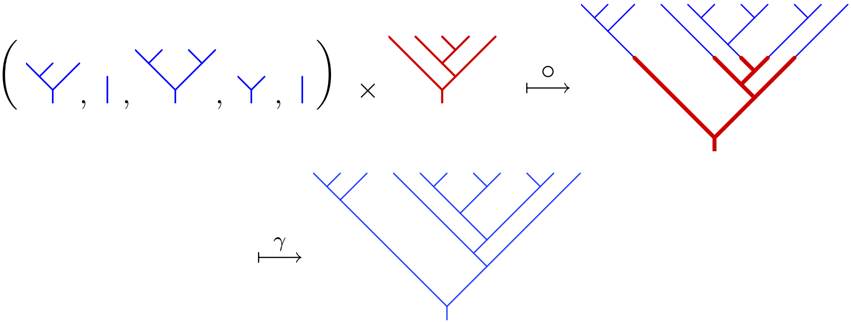


and grafting:



Loday–Ronco Hopf algebra.

The n^{th} component of $\mathcal{Y}Sym$ has basis the collection of binary trees with n interior nodes, and thus $n+1$ leaves, denoted $\mathcal{Y}_n$.

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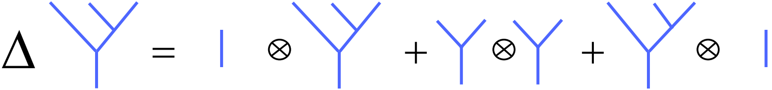

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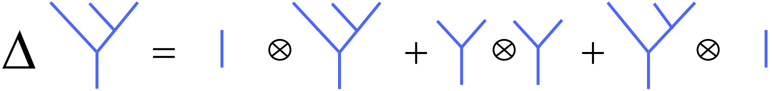
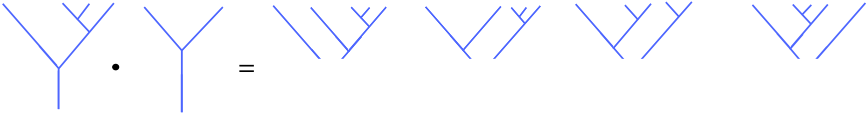


Here is how to multiply two trees:



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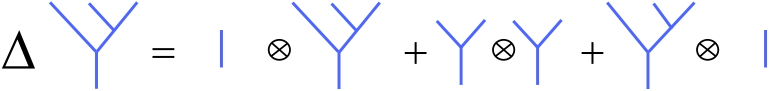

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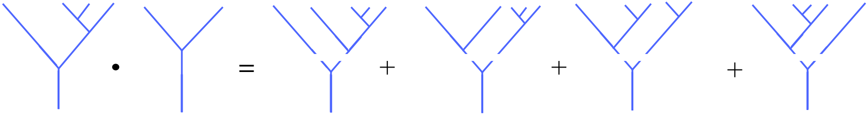

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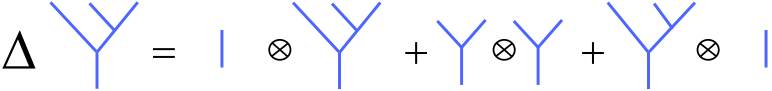

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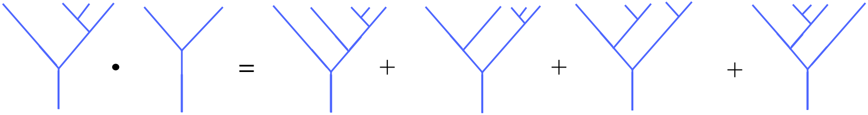


Our cast as species.

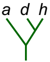
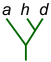
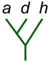
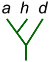
Species.

A *species* is an endofunctor of Finite Sets with bijections.

- *Example*: The species $\mathcal{L}$ of lists takes a set to linear orders of that set.

$$\mathcal{L}(\{a, d, h\}) = \{\ {\scriptstyle a<d<h,\ a<h<d,\ h<a<d,\ h<d<a,\ d<a<h,\ d<h<a}\ \}$$

- *Example*: The species $\mathcal{Y}$ of binary trees takes a set to trees with labeled leaves.

$\mathcal{Y}(\{a, d, h\}) = \{$  ,  , ... ,  ,  , ...$\}$

Species composition.

We define the composition of two species:

$$(\mathcal{G} \circ \mathcal{H})(U) = \bigsqcup_{\pi} \mathcal{G}(\pi) \times \prod_{U_i \in \pi} \mathcal{H}(U_i)$$

where the union is over partitions of U into any number of nonempty disjoint parts.

$$\pi = \{U_1, U_2, \ldots, U_n\} \text{ such that } U_1 \sqcup \cdots \sqcup U_n = U.$$

Familiar(?): also known as the cumulant formula, and the moment sequence of a random variable, and the domain for operad composition:

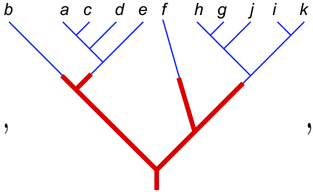
$$\gamma : \mathcal{F} \circ \mathcal{F} \to \mathcal{F}$$

Ordered tree of trees: indelible grafting.

Example:

$$(\mathfrak{S} \circ \mathcal{Y})(\{a, b, c, d, e, f, g, h, i, j, k\}) =$$

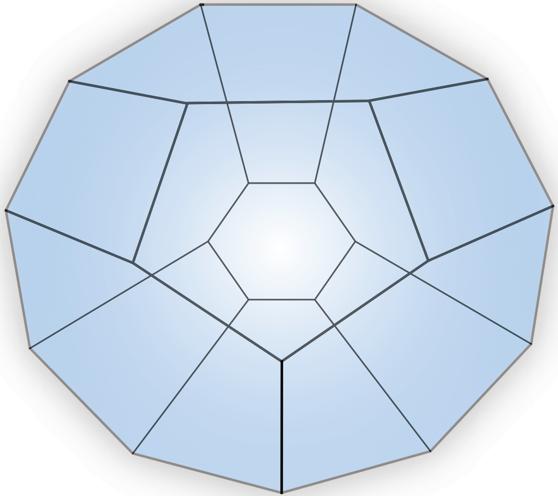
$\Bigg\{ \ldots,$



$, \ldots \Bigg\} = \left\{ \frac{\text{labeled trees}}{\text{ordered tree}} \right\}$

The graft is indelible! We will focus on the structure type, forgetting the labels.

Example: $\mathfrak{S} \circ \mathcal{Y}$ in 3d.



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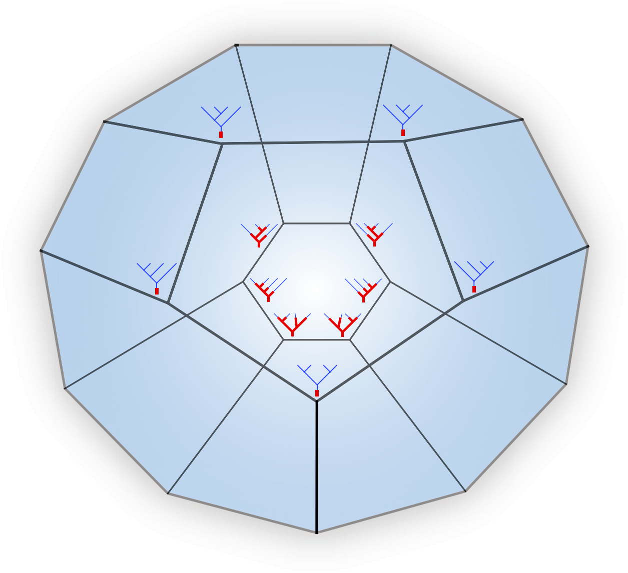

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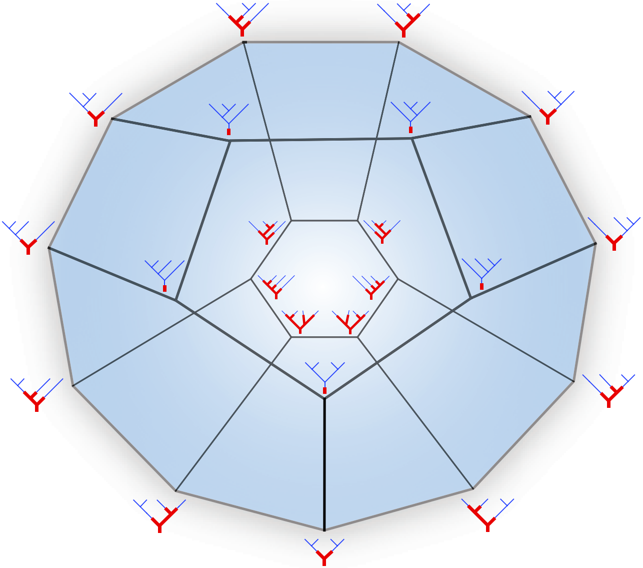


Composition of coalgebras

Given two graded coalgebras we combine them in a way reminiscent of species composition.

Let $\mathcal{C}$ and $\mathcal{D}$ be two graded coalgebras. We will form a new coalgebra $\mathcal{E} = \mathcal{D} \circ \mathcal{C}$ on the vector space

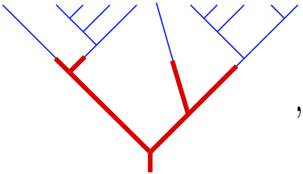
$$\mathcal{D} \circ \mathcal{C} \ := \ \bigoplus_{n \geq 0} \mathcal{D}_n \otimes \mathcal{C}^{\otimes(n+1)} \,.$$

Example

By construction, the basis for a composition of coalgebras is indexed by the types of the composition of the species.

$$\mathfrak{S}Sym \circ \mathcal{Y}Sym =$$

span $\Big\{ \ldots,$



$, \ldots \Big\}$ = span $\left\{ \dfrac{\text{trees}}{\text{ordered tree}} \right\}$

Finally: results.

Results:

Given composed coalgebras $\mathcal{E} = \mathcal{C} \circ \mathcal{D}$,

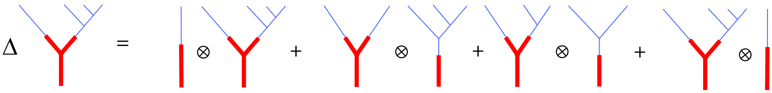
Theorem If $\mathcal{C}$ and $\mathcal{D}$ are cofree coalgebras then so is $\mathcal{E}$. Primitives are easy to compute.

Theorem If either $\mathcal{C}$ or $\mathcal{D}$ is a Hopf algebra with a special connection to $\mathcal{E}$, then $\mathcal{E}$ is a (one sided) Hopf algebra too. Antipodes are found recursively.

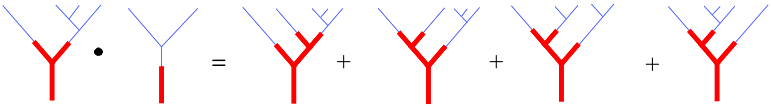
Conj. If $\mathcal{C}_n$ and $\mathcal{D}_n$ index vertices of polytopes, so does $\mathcal{E}_n$.

Examples from trees.

Here is an example of the coproduct in $\mathcal{Y}Sym \circ \mathcal{Y}Sym$:



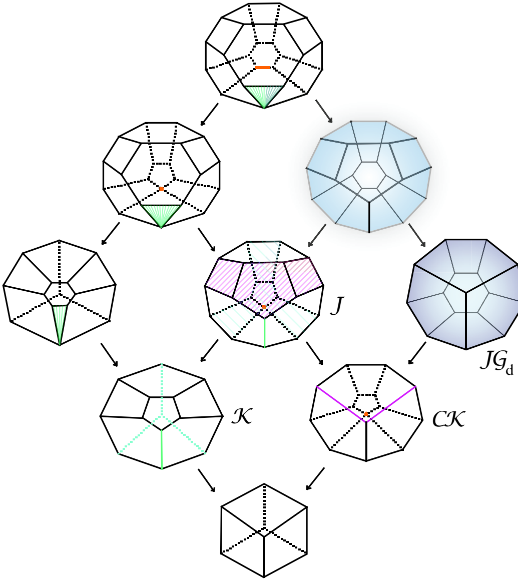
Here is an example of the product in $\mathcal{Y}Sym \circ \mathcal{Y}Sym$:

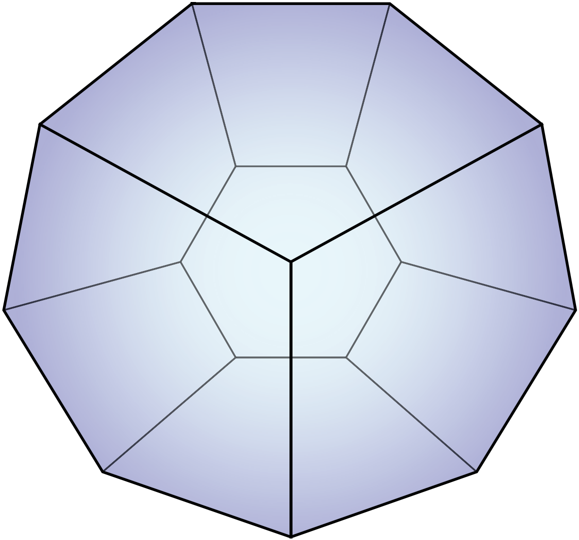


Composing species of trees.

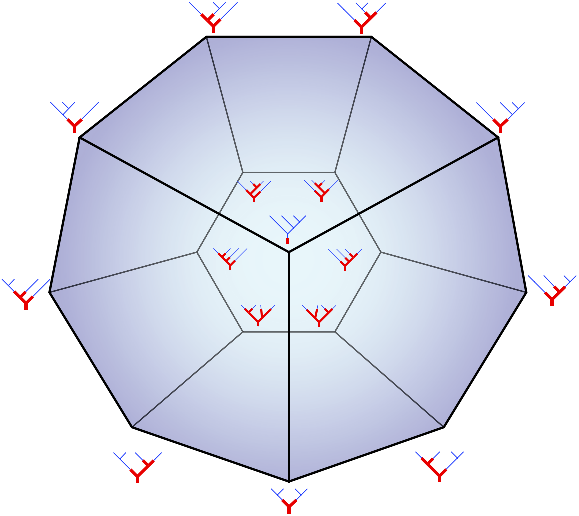
$$
\begin{tikzcd}[column sep=small]
 & & \mathfrak{S}\circ\mathfrak{S} \arrow[dl] \arrow[dr] & & \\
 & \mathcal{Y}\circ\mathfrak{S} \arrow[dl] \arrow[dr] & & \mathfrak{S}\circ\mathcal{Y} \arrow[dl] \arrow[dr] & \\
\mathfrak{C}\circ\mathfrak{S} \arrow[dr] & & \mathcal{Y}\circ\mathcal{Y} \arrow[dl] \arrow[dr] & & \mathfrak{S}\circ\mathfrak{C} \arrow[dl] \\
 & \mathfrak{C}\circ\mathcal{Y} \arrow[dr] & & \mathcal{Y}\circ\mathfrak{C} \arrow[dl] & \\
 & & \mathfrak{C}\circ\mathfrak{C} & &
\end{tikzcd}
$$

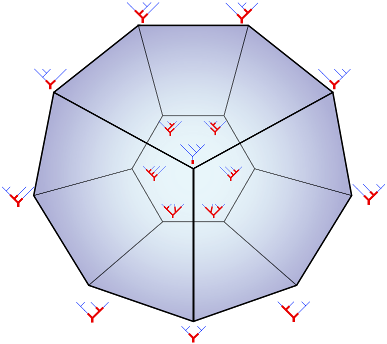
Polytope conjecture.





$\mathfrak{S} \circ \mathfrak{C}$





This polytope has been seen before!
Stellohedron (S. Devadoss, A. Postnikov, V. Reiner, L. Williams).

$$\text{Number of vertices } = \sum_{k=0}^{n} \frac{n!}{k!}$$

Thanks!

Questions and comments?