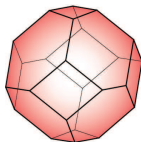


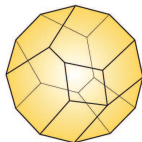
Composing species and composing coalgebras.

Stefan Forcey, U. Akron
Aaron Lauve, Loyola U. Chicago
Frank Sottile, Texas A&M U.

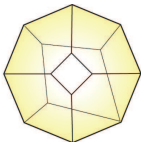
September 8, 2011



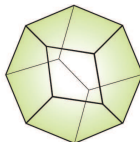
Permutohedron



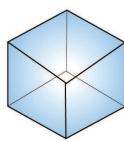
Multiplihedron



Composihedron



Associahedron



Hypercube

“Niceness is hereditary in species.”

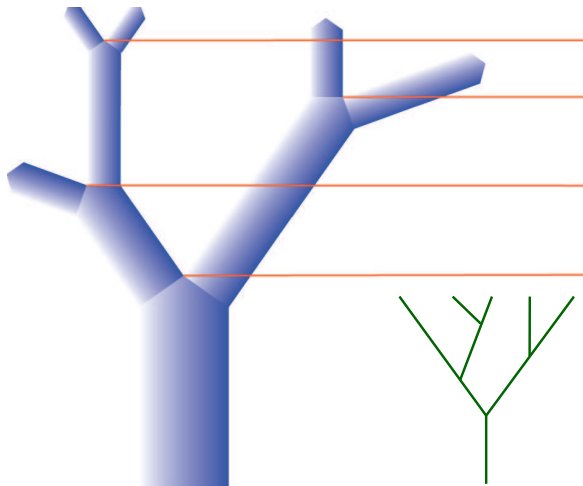
“Niceness is hereditary in species.”

For this talk, nice properties of species of coalgebras will be:

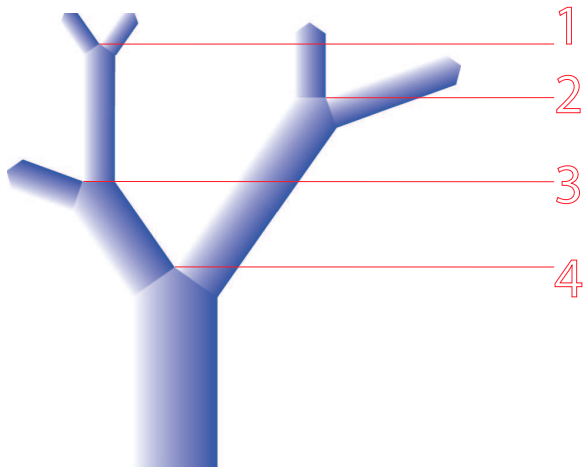
1. Cofree-ness,
2. Hopf-ness,
3. Polytopal-ness.

But first, our cast of characters:

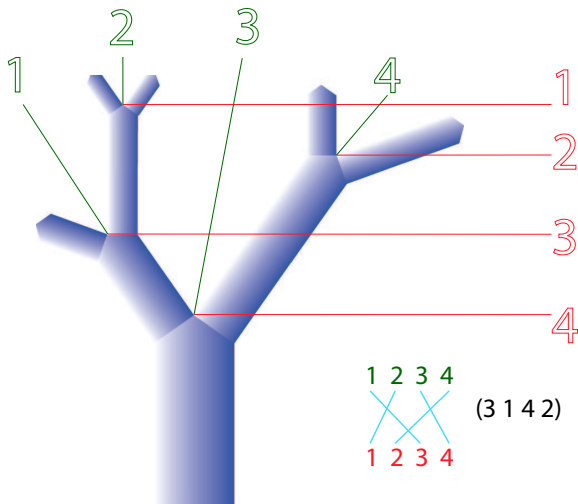
Ordered trees \mathfrak{S} .



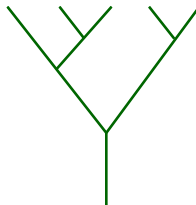
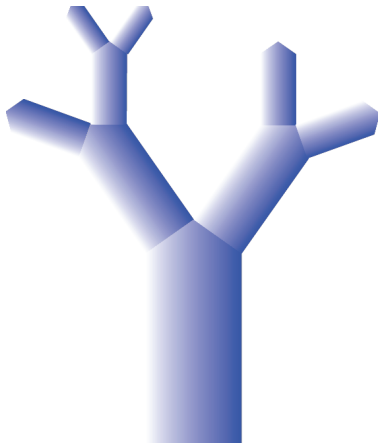
Ordered trees \mathfrak{S} .



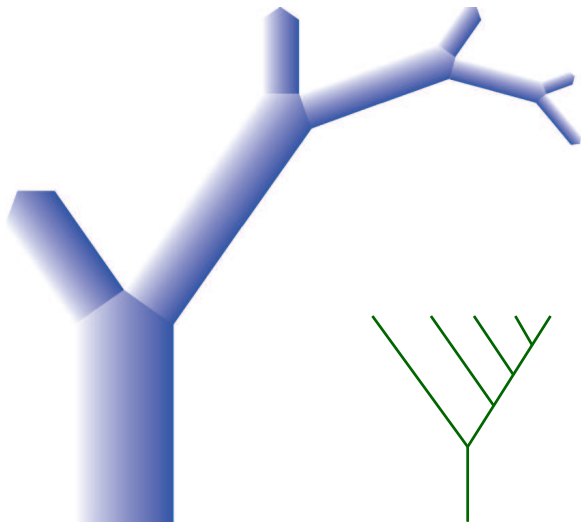
Ordered trees are permutations \mathfrak{S}_n .



Binary trees \mathcal{Y} .

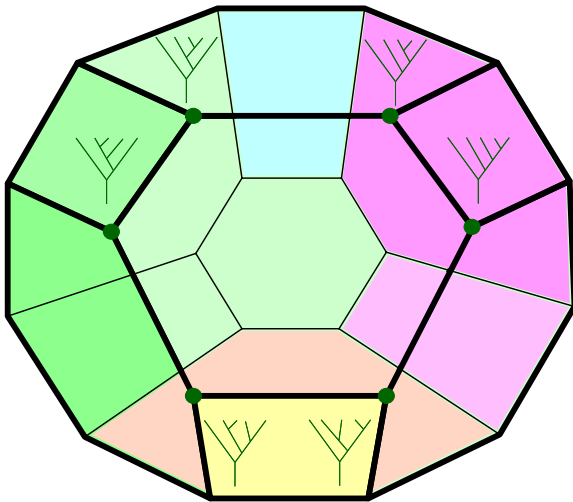


Combed binary trees \mathfrak{C} .

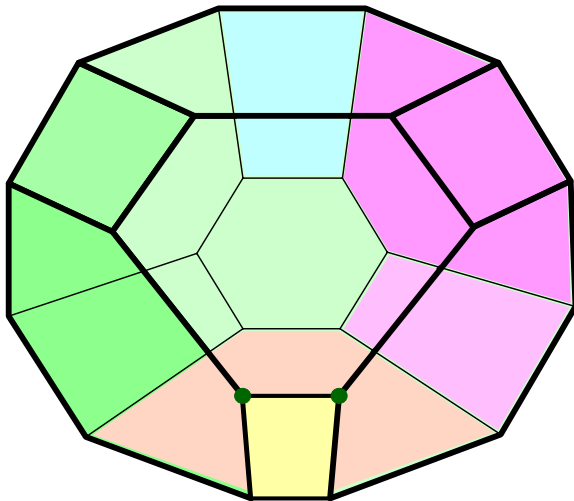


Our cast as Polytopes.

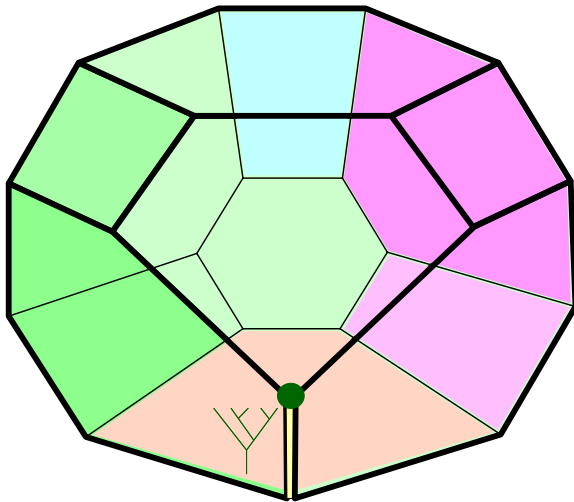
\mathfrak{S} : Permutohedron.



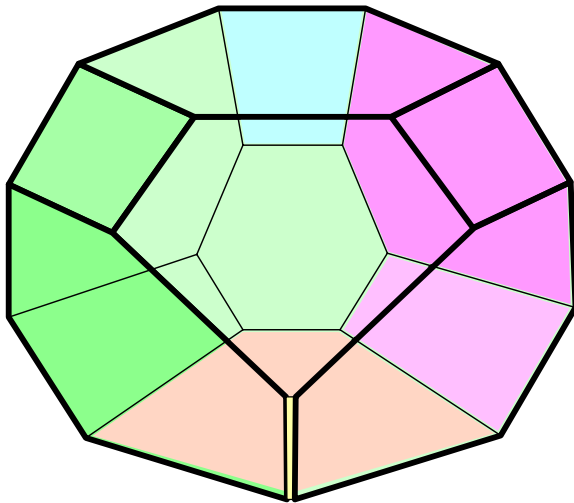
Tonks cellular projection.



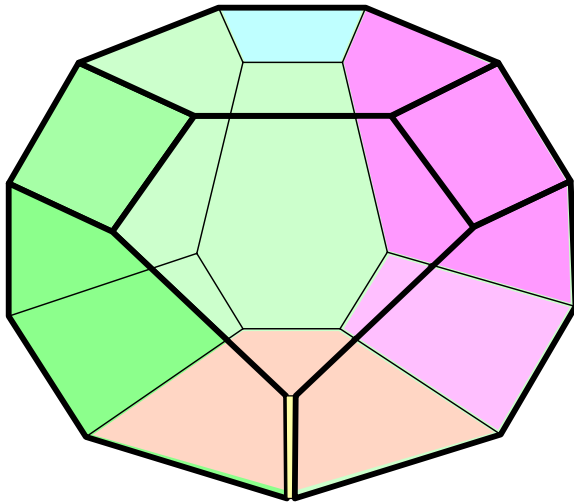
Tonks cellular projection.



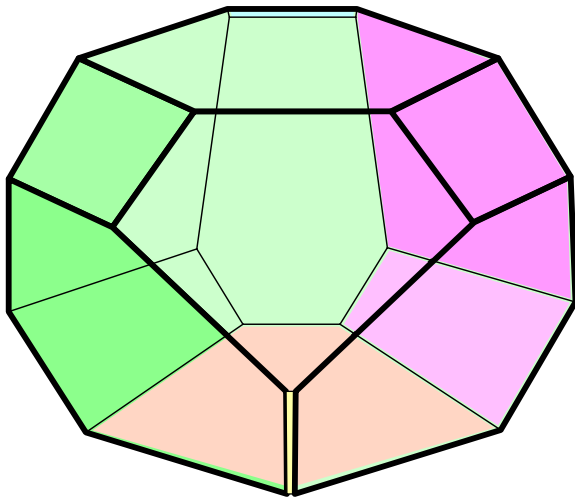
Tonks cellular projection.



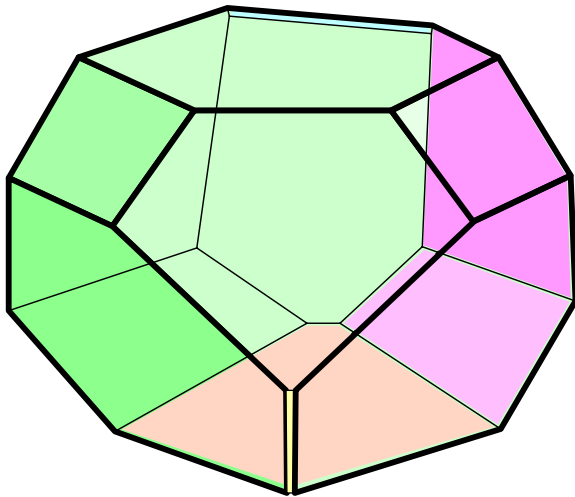
Tonks cellular projection.



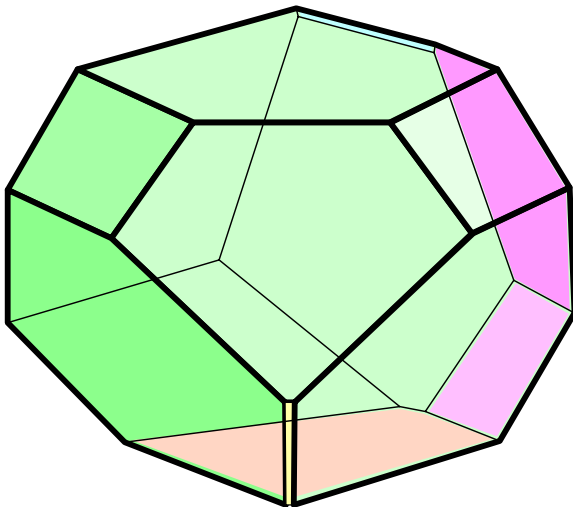
Tonks cellular projection.



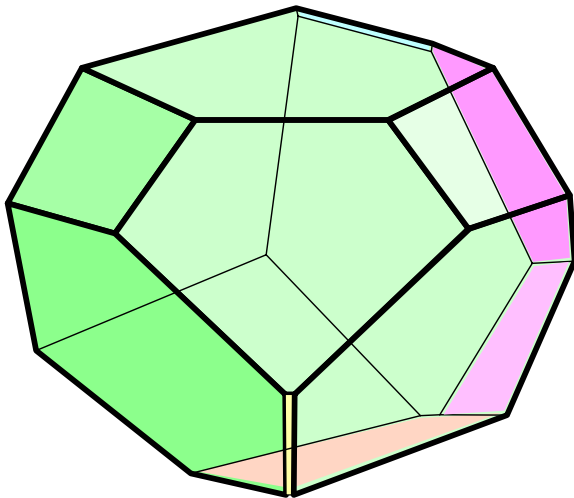
Tonks cellular projection.



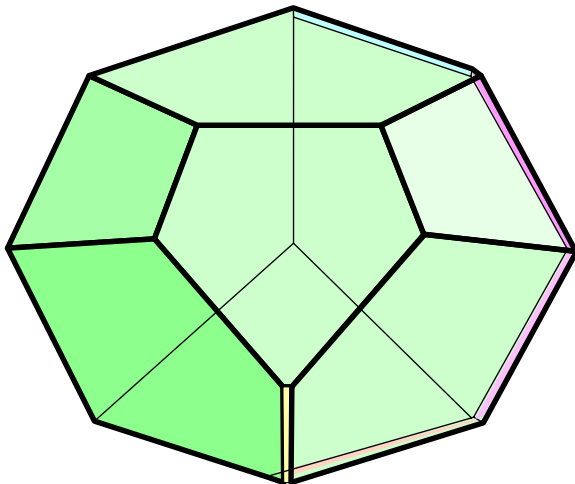
Tonks cellular projection.



Tonks cellular projection.



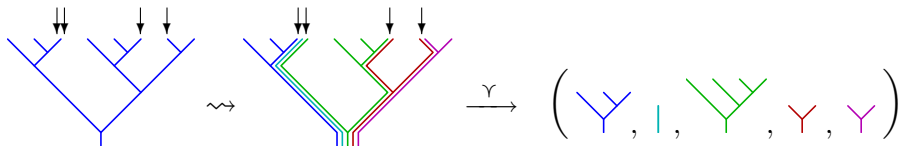
\mathcal{Y} : Associahedron



Our cast as graded Hopf algebras.

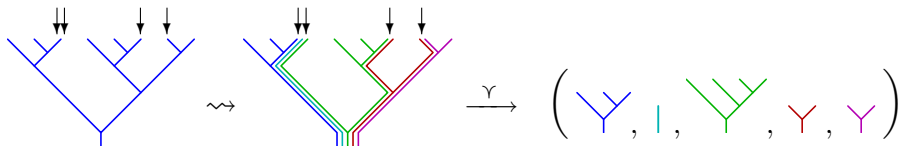
A Hopf algebra of binary trees.

Two operations on trees: splitting

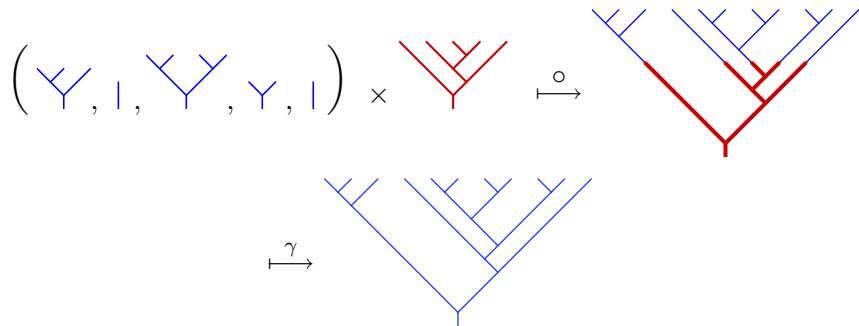


A Hopf algebra of binary trees.

Two operations on trees: splitting



and grafting:



Loday–Ronco Hopf algebra.

The n^{th} component of $\mathcal{Y}Sym$ has basis the collection of binary trees with n interior nodes, and thus $n + 1$ leaves, denoted \mathcal{Y}_n .

Loday–Ronco Hopf algebra.

The n^{th} component of $\mathcal{Y}Sym$ has basis the collection of binary trees with n interior nodes, and thus $n + 1$ leaves, denoted \mathcal{Y}_n .

Here is the coproduct:

$$\Delta \left(\text{tree}_1 \right) = \left| \right| \otimes \text{tree}_2 + \text{tree}_3 \otimes \text{tree}_4 + \text{tree}_5 \otimes \left| \right|$$

The diagram shows the coproduct of the coproduct tree (a tree with 2 interior nodes and 3 leaves). The result is the sum of three terms, each representing a way to cut the tree into two parts. The first term is a vertical line (representing a tree with 0 interior nodes and 1 leaf) tensored with the coproduct tree. The second term is a tree with 1 interior node and 2 leaves tensored with a tree with 1 interior node and 2 leaves. The third term is the coproduct tree tensored with a vertical line.

Loday–Ronco Hopf algebra.

The n^{th} component of $\mathcal{Y}\text{Sym}$ has basis the collection of binary trees with n interior nodes, and thus $n + 1$ leaves, denoted \mathcal{Y}_n .

Here is the coproduct:

$$\Delta \text{ (tree)} = \text{leaf} \otimes \text{tree} + \text{tree} \otimes \text{leaf}$$

Here is how to multiply two trees:

$$\text{tree}_1 \cdot \text{tree}_2 = \text{tree}_1 \text{ (left child of tree}_2) + \text{tree}_1 \text{ (right child of tree}_2) + \text{tree}_1 \text{ (root of tree}_2)$$

Loday–Ronco Hopf algebra.

The n^{th} component of $\mathcal{Y}\text{Sym}$ has basis the collection of binary trees with n interior nodes, and thus $n + 1$ leaves, denoted \mathcal{Y}_n .

Here is the coproduct:

$$\Delta \text{ (tree with 2 interior nodes) } = | \otimes \text{ (tree with 2 interior nodes) } + \text{ (tree with 1 interior node) } \otimes \text{ (tree with 1 interior node) } + \text{ (tree with 2 interior nodes) } \otimes |$$

Here is how to multiply two trees:

$$\text{ (tree with 2 interior nodes) } \cdot \text{ (tree with 1 interior node) } = \text{ (tree with 3 interior nodes) } + \text{ (tree with 3 interior nodes) } + \text{ (tree with 3 interior nodes) } + \text{ (tree with 3 interior nodes) }$$

Loday–Ronco Hopf algebra.

The n^{th} component of $\mathcal{Y}\text{Sym}$ has basis the collection of binary trees with n interior nodes, and thus $n + 1$ leaves, denoted \mathcal{Y}_n .

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Here is how to multiply two trees:

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Our cast as species.

Species.

A *species* is an endofunctor of Finite Sets with bijections.

- *Example:* The species \mathcal{L} of lists takes a set to linear orders of that set.

$$\mathcal{L}(\{a, d, h\}) = \{ a < d < h, a < h < d, h < a < d, h < d < a, d < a < h, d < h < a \}$$

- *Example:* The species \mathcal{Y} of binary trees takes a set to trees with labeled leaves.

$$\mathcal{Y}(\{a, d, h\}) = \{ \begin{array}{c} a \quad d \quad h \\ \diagdown \quad \diagup \\ \text{Y} \end{array}, \begin{array}{c} a \quad h \quad d \\ \diagdown \quad \diagup \\ \text{Y} \end{array}, \dots, \begin{array}{c} a \quad d \quad h \\ \diagdown \quad \diagup \\ \text{Y} \end{array}, \begin{array}{c} a \quad h \quad d \\ \diagdown \quad \diagup \\ \text{Y} \end{array}, \dots \}$$

Species composition.

We define the composition of two species:

$$(\mathcal{G} \circ \mathcal{H})(U) = \bigsqcup_{\pi} \mathcal{G}(\pi) \times \prod_{U_i \in \pi} \mathcal{H}(U_i)$$

where the union is over partitions of U into any number of nonempty disjoint parts.

$$\pi = \{U_1, U_2, \dots, U_n\} \text{ such that } U_1 \sqcup \dots \sqcup U_n = U.$$

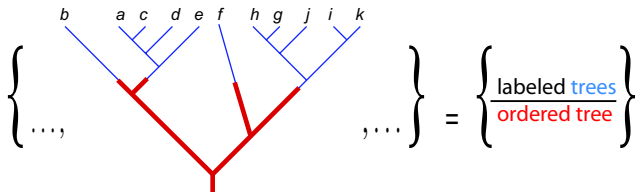
Familiar(?): also known as the cumulant formula, and the moment sequence of a random variable, and the domain for operad composition:

$$\gamma : \mathcal{F} \circ \mathcal{F} \rightarrow \mathcal{F}$$

Ordered tree of trees: indelible grafting.

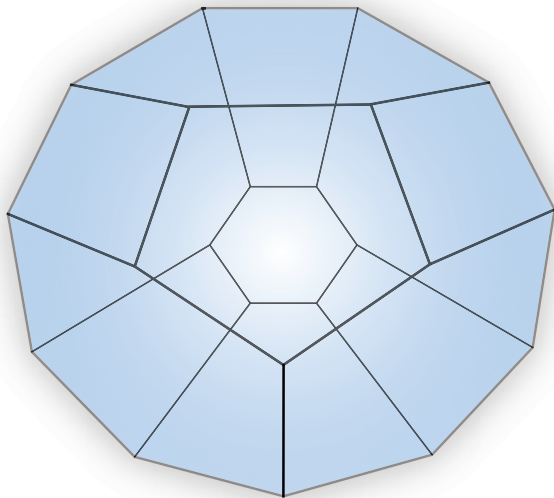
Example:

$$(\mathfrak{S} \circ \mathcal{Y})(\{a, b, c, d, e, f, g, h, i, j, k\}) =$$

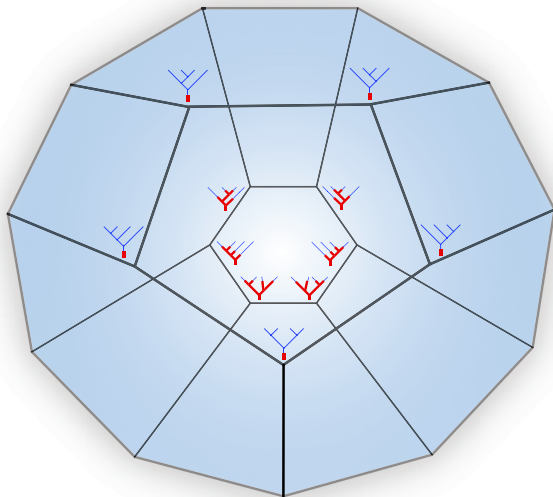


The graft is indelible! We will focus on the structure type, forgetting the labels.

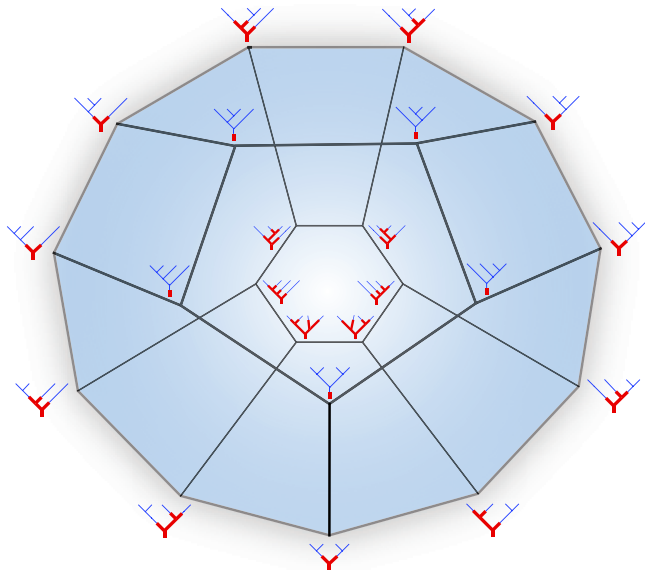
Example: $\mathfrak{S} \circ \mathcal{Y}$ in 3d.



Example: $\mathfrak{S} \circ \mathcal{Y}$ in 3d.



Example: $\mathfrak{S} \circ \mathcal{Y}$ in 3d.



Composition of coalgebras

Given two graded coalgebras we combine them in a way reminiscent of species composition.

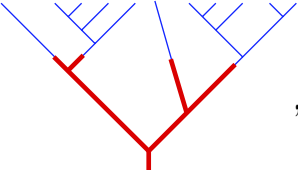
Let \mathcal{C} and \mathcal{D} be two graded coalgebras. We will form a new coalgebra $\mathcal{E} = \mathcal{D} \circ \mathcal{C}$ on the vector space

$$\mathcal{D} \circ \mathcal{C} := \bigoplus_{n \geq 0} \mathcal{D}_n \otimes \mathcal{C}^{\otimes (n+1)}.$$

Example

By construction, the basis for a composition of coalgebras is indexed by the types of the composition of the species.

$$\mathfrak{S}Sym \circ \mathcal{Y}Sym =$$

$$\text{span} \left\{ \dots, \text{  , \dots \right\} = \text{span} \left\{ \frac{\text{trees}}{\text{ordered tree}} \right\}$$

Finally: results.

Results:

Given composed coalgebras $\mathcal{E} = \mathcal{C} \circ \mathcal{D}$,

Theorem If \mathcal{C} and \mathcal{D} are cofree coalgebras then so is \mathcal{E} . Primitives are easy to compute.

Theorem If either \mathcal{C} or \mathcal{D} is a Hopf algebra with a special connection to \mathcal{E} , then \mathcal{E} is a (one sided) Hopf algebra too. Antipodes are found recursively.

Conj. If \mathcal{C}_n and \mathcal{D}_n index vertices of polytopes, so does \mathcal{E}_n .

Examples from trees.

Here is an example of the coproduct in $\mathcal{YSym} \circ \mathcal{YSym}$:

$$\Delta \left(\text{Diagram 1} \right) = \left(\text{Diagram 2} \right) \otimes \left(\text{Diagram 3} \right) + \left(\text{Diagram 4} \right) \otimes \left(\text{Diagram 5} \right) + \left(\text{Diagram 6} \right) \otimes \left(\text{Diagram 7} \right) + \left(\text{Diagram 8} \right) \otimes \left(\text{Diagram 9} \right)$$

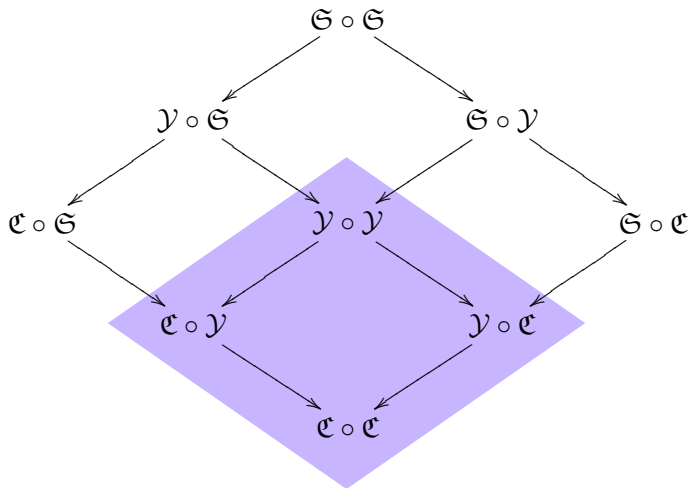
The diagram shows the coproduct Δ applied to a tree with 5 internal nodes (red) and 4 external nodes (blue). The result is a sum of four tensor products of trees. In each tensor product, one of the original tree's internal nodes is isolated as a single vertical red line, while the rest of the tree structure remains.

Here is an example of the product in $\mathcal{YSym} \circ \mathcal{YSym}$:

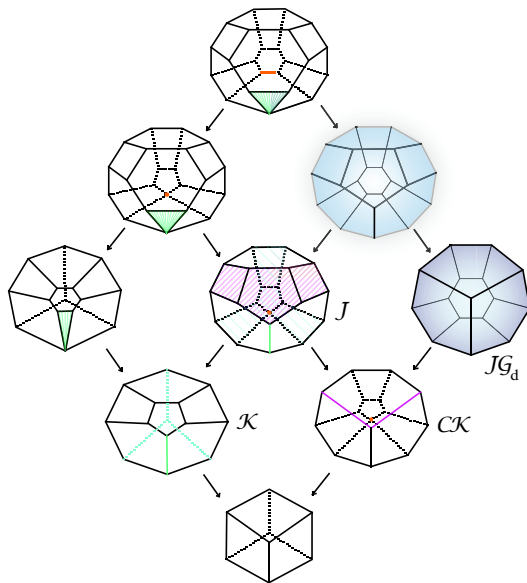
$$\left(\text{Diagram 1} \right) \cdot \left(\text{Diagram 2} \right) = \left(\text{Diagram 3} \right) + \left(\text{Diagram 4} \right) + \left(\text{Diagram 5} \right) + \left(\text{Diagram 6} \right)$$

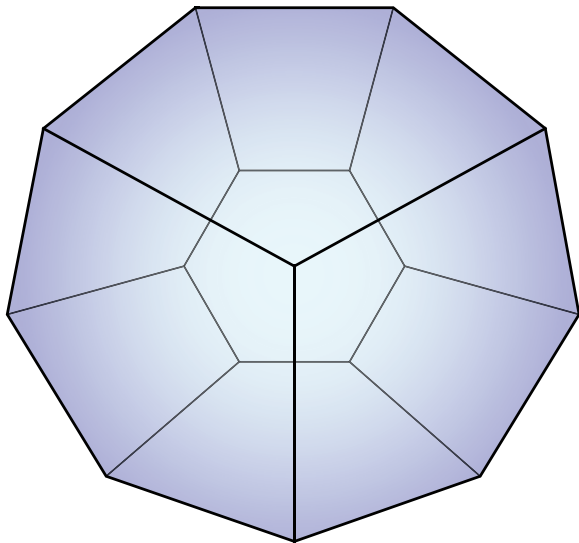
The diagram shows the product \cdot applied to the same tree as in the coproduct example and a tree consisting of a single vertical red line. The result is a sum of four trees, each representing a different way to graft the single vertical line onto the original tree structure.

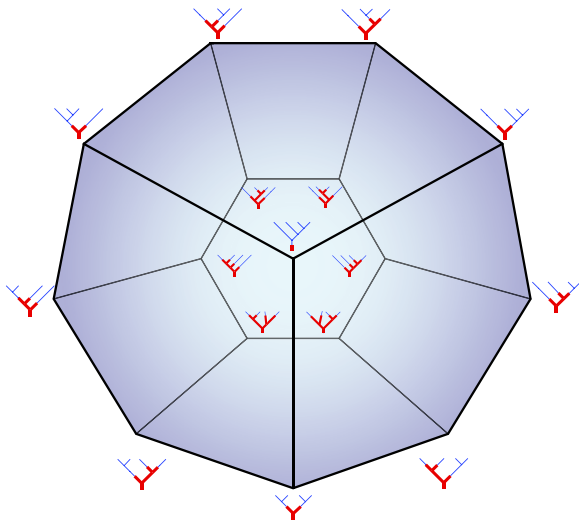
Composing species of trees.

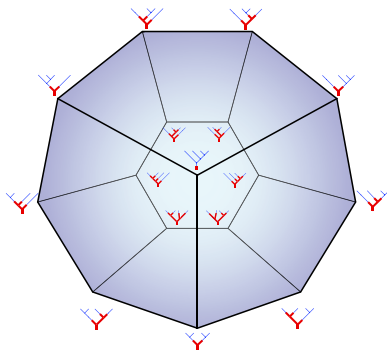


Polytope conjecture.









This polytope has been seen before!
Stellohedron (S. Devadoss, A. Postnikov, V. Reiner, L. Williams).

$$\text{Number of vertices} = \sum_{k=0}^n \frac{n!}{k!}$$

Thanks!

Advertisement:

<http://www.math.uakron.edu/~sf34/hedra.htm>

Questions and comments?