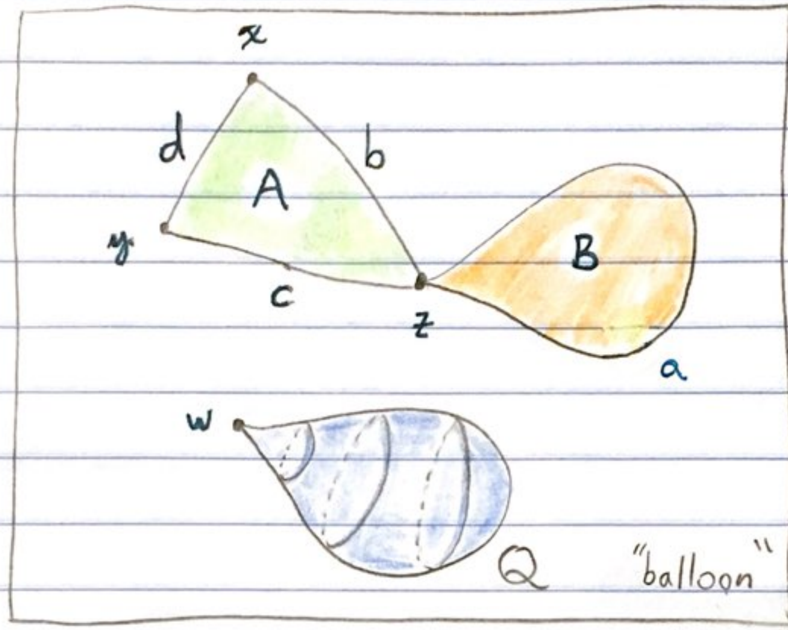
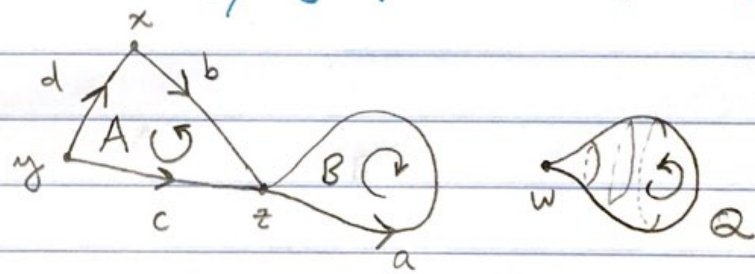
Ex:

S =



Find the homology groups (cellular) $H_n(S)$

① Orient:



② Boundary: (on cells)

0-cells: $\partial_0(x) = \partial(y) = \partial(z) = \partial(w) = 0$

1-cells: $\partial_1(a) = z - z = 0$

$\partial_1(b) = z - x$

$\partial_1(c) = z - y$

$\partial_1(d) = x - y$

2-cells: $\partial_2(A) = c - b - d$

$\partial_2(B) = -a$

$\partial_2(Q) = 0$

(3) Image (Range) and Kernel (Null-space) of ∂_n

$$\cdots \to \underbrace{0}_{C_3} \xrightarrow{\partial_3} \underbrace{\mathrm{span}\{A, B, Q\}}_{C_2} \xrightarrow{\partial_2} \underbrace{\mathrm{Span}\{a, b, c, d\}}_{C_1} \xrightarrow{\partial_1} \underbrace{\mathrm{span}\{x, y, z, w\}}_{C_0} \xrightarrow{\partial_0} 0.$$

$$\partial_1 = \begin{array}{cc} \begin{array}{cccc} a & b & c & d \end{array} & \\ \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} x \\ y \\ z \\ w \end{array} \end{array} \to \left[\begin{array}{cccc|c} 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R1 \leftarrow -R1$, $R3 \leftarrow R3 - R1$, $R2 \leftarrow -R2$, $R3 \leftarrow R3 - R2$

$$\sim \left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{l} k_2 - k_4 = 0 \\ k_3 + k_4 = 0 \end{array} \Rightarrow \begin{cases} k_2 = k_4 \\ k_3 = -k_4 \\ k_4 = \text{free} \\ k_1 = \text{free} \end{cases}$$

$$\Rightarrow \ker \partial_1 = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$= \mathrm{span}\{a, \ b - c + d\}$$

$$\Rightarrow \mathrm{Im}\, \partial_1 = \mathrm{span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$= \mathrm{span}\{-x + z, \ -y + z\}$$

(4)

$$H_0 = \ker \partial_0 / \mathrm{Im}\, \partial_1 = \mathrm{span}\{x, y, z, w\} / \mathrm{span}\{z - x, z - y\}$$

$$= \langle x, y, z, w \mid z = x, \ z = y \rangle$$

$$= \langle x, w \mid \emptyset \rangle$$

(3)

$$\partial_2 = \begin{array}{c} \begin{array}{ccc} A & B & Q \end{array} \\ \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \Rightarrow \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right]$$

$R1 \leftrightarrow R3$, $R2 \leftarrow R2 + R1$, $R4 \leftarrow R4 + R1$, $R3 \leftarrow -R3$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow k_1 = 0,\ k_2 = 0,\ k_3 = \text{free}$

$$\ker \partial_2 = \text{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} = \text{span}\{Q\}$$

$$\text{Im } \partial_2 = \text{span}\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}$$

$$= \text{span}\{-b + c + -d, -a\}$$

(4)

$$H_1 = \ker \partial_1 / \text{Im } \partial_2 = \text{span}\{a, b-c+d\} / \text{span}\{-b+c-d, -a\}$$

$$= \text{span}\{a, b-c+d\} / \text{span}\{a, b-c+d\}$$

$$= \langle a, q \mid a = 0, q = 0 \rangle$$

$$= \langle 0 \rangle$$

$$H_2 = \ker \partial_2 / \text{Im } \partial_3 = \text{span}\{Q\} / \text{span}\{0\}$$

$$= \langle Q \mid \emptyset \rangle = \mathbb{Z}$$

$$\boxed{H_0 = \langle x, w \mid \emptyset \rangle \quad H_1 = \langle 0 \rangle, \quad H_2 = \langle Q \mid \emptyset \rangle}$$

$H_0 = \mathbb{Z} \oplus \mathbb{Z}$, $H_1 = 0$, $H_2 = \mathbb{Z}$

$$\to 0 \xrightarrow{\partial_3} \text{span}\{A, B, Q\} \xrightarrow{\partial_2} \text{span}\{a, b, c, d\} \xrightarrow{\partial_1} \text{span}\{x, y, z, w\} \xrightarrow{\partial_0} 0$$

Alt.

$$\begin{aligned} H_0 = \ker \partial_0 / \operatorname{Im} \partial_1 &= \text{span}\{x, y, z, w\} / \text{span}\{z-x, z-y, x-y\} \\ &= \langle x, y, z, w \mid x = z,\ z = y,\ x = y \rangle \\ &= \langle x, w \mid \emptyset \rangle \\ &= \mathbb{Z} \oplus \mathbb{Z} \end{aligned}$$

(ordered pairs of integers)