Example $S = P^2$

disks, cells, chains, complexes, cycles, boundaries ...

... kernels, images, quotients, groups, modules

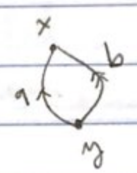
0-skeleton

$x \cdot$

$y \cdot$

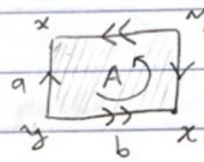
0-disks = 0-cells

1-skeleton



1-disks = 1-cells

Cell complex



2-disks = 2-cells

$\emptyset$ ...

3-disks ... = 3-cells

For $S = P^2$

Chain Complex free abelian chain groups

$$\cdots 0 \xrightarrow{\partial_4} C_3 = \langle 0 \rangle \xrightarrow{\partial_3} C_2 = \langle A \rangle \xrightarrow{\partial_2} C_1 = \langle a, b \rangle \xrightarrow{\partial_1} C_0 = \langle x, y \rangle \xrightarrow{\partial_0 = 0} 0$$

$\partial_4(0) = 0$

$\partial_3(0) = 0$

$C_2 = \{A, -A, 2A = A + A\} \ldots$

$C_1 = \{a, -a, b, -b, 2a - 3b\} \ldots$

$C_0 = \{x, y, 3x, -5y, x + 2y\} \ldots$

just generators, no relations

$\partial_2(A) = b - a + b - a = 2(b-a)$

If you know the output of generators, the map is known!

$\partial_1(a) = x - y$

$\partial_1(b) = x - y$

Homology groups

$H_3 = \ker \partial_3 / \operatorname{Im} \partial_4 = \boxed{\langle 0 \rangle / \langle 0 \rangle}$ $\boxed{= \langle 0 \rangle}$

Here the generators are top relations are bottom (written that way for abelian)

$H_2 = \ker \partial_2 / \operatorname{Im} \partial_3 = \boxed{\langle 0 \rangle / \langle 0 \rangle} = \boxed{\langle 0 \rangle}$

$H_1 = \ker \partial_1 / \operatorname{Im} \partial_2$

$= \text{1-cycles} / \text{1-boundaries}$

$= \boxed{\langle b - a \rangle / \langle 2(b-a) \rangle}$

Let $q = b - a$

$\cong \langle q \mid 2q = 0 \rangle$

$= \langle q \mid q = -q \rangle$

... and this is familiar: $\boxed{H_1 \cong \mathbb{Z}/2\mathbb{Z}}$ the integers mod 2

Note: $\langle p, q \rangle = \text{span}\{p, q\}$.

Note: if needed we can use linear alg. to find kernel (null space) and image (range).

$$\text{span}\{a,b\} \xrightarrow{\partial_1} \text{span}\{x,y\} \qquad \left.\begin{aligned} \partial_1(a) &= x - y \\ \partial_1(b) &= x - y \end{aligned}\right\} \quad \partial_1 = \begin{array}{cc} a & b \\ \left[\begin{matrix} 1 & 1 \\ -1 & -1 \end{matrix}\right. & \left.\right] \end{array}$$

$$\partial_1 = \begin{array}{cc} a & b \end{array} \atop \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Find kernel: ∂_1

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow 1c_1 + 1c_2 = 0$

$\Rightarrow c_1 = -c_2$

$\Rightarrow \ker \partial_1 = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$

$= \boxed{\text{span}\{-a+b\}}$

Find range: ∂_1 — pivot colum above in r.r.e.f. is first column, so

$\text{Im}\, \partial_1 = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} = \boxed{\text{span}\{x-y\}}$

$$\text{span}\{A\} \xrightarrow{\partial_2} \text{span}\{a,b\}$$

Find ker ∂_2: $\left\{ \partial_2(A) = 2b - 2a \right\}$ $\partial_2 = \begin{array}{c} A \\ \begin{bmatrix} -2 \\ 2 \end{bmatrix} \end{array}$

$$\ker \partial_2: \left[\begin{array}{c|c} -2 & 0 \\ 2 & 0 \end{array}\right] \sim \left[\begin{array}{c|c} -2 & 0 \\ 0 & 0 \end{array}\right] \sim \left[\begin{array}{c|c} 1 & 0 \\ 0 & 0 \end{array}\right]$$

$\Rightarrow c_1 = 0$

$\Rightarrow \ker \partial_2 = \boxed{0}$

range ∂_2: $\text{Im}\, \partial_2 = \text{span}\left\{\begin{pmatrix} -2 \\ 2 \end{pmatrix}\right\} = \text{span}\{-2a+2b\}$

$= \boxed{\text{span}\{2(b-a)\}}$

$$H_0 = \ker \partial_0 \Big/ \operatorname{Im} \partial_1$$

$$= \text{0-cycles} \Big/ \text{0-bdy's}$$

$$= \operatorname{span}\{x, y\} \Big/ \operatorname{span}\{x-y\}$$

$$= \langle x, y \rangle \Big/ \langle x-y \rangle$$

$$= \langle x, y \mid x-y=0 \rangle$$

$$= \langle x, y \mid x=y \rangle$$

$$= \langle x \rangle$$

$$\cong \mathbb{Z}$$

$$\boxed{\begin{aligned} H_3(P^2) &= 0 \\ H_2(P^2) &= 0 \\ H_1(P^2) &= \mathbb{Z}/2\mathbb{Z} \\ H_0(P^2) &= \mathbb{Z} \end{aligned}}$$

Now, if we let coeffizients be $\mathbb{R}$ instead of $\mathbb{Z}$, every thing is the same up to final calculations.

$$H_3 = H_2 = \boxed{0}$$

$$H_1 = \operatorname{span}\{b-a\} \Big/ \operatorname{span}\{2(b-a)\} = \operatorname{span}\{q\} \Big/ \operatorname{span}\{2q\}$$

$$= \mathbb{R}/\mathbb{R} = \boxed{0}$$

$$H_0 = \operatorname{span}\{x, y\} \Big/ \operatorname{span}\{x-y\} = \mathbb{R}^2/\mathbb{R} = \boxed{\mathbb{R}}$$