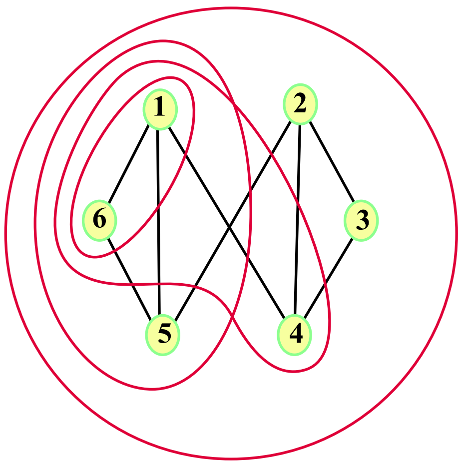
Dr. Forcey, Step 1

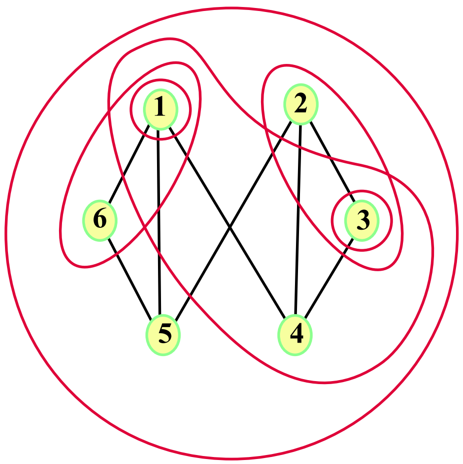
$\mathscr{B}1$



$\mathscr{B}1 = \{ X, \{1, 6\}, \{1, 5, 6\}, \{1, 4, 6\}\}$

$\mathscr{O}1 = \mathscr{B}1 \cup \{\{\}, \{1, 4, 5, 6\}\}$

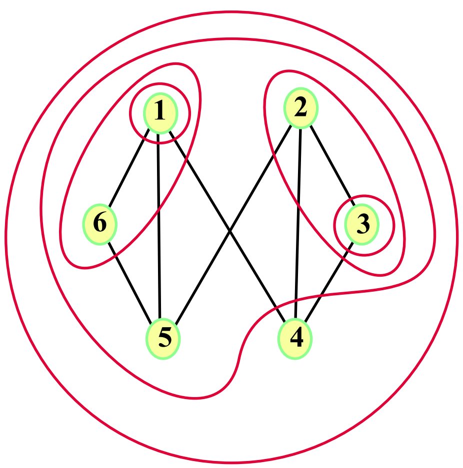
$\mathscr{B}2$



$\mathscr{B}2 = \{ X, \{1\}, \{3\}, \{1, 6\}, \{2, 3\}, \{1, 3, 4\}\}$

$\mathscr{O}2 = \mathscr{B}2 \cup \{\{\}, \{1, 3\}, \{1, 2, 3\}, \{1, 3, 6\}, \{1, 2, 3, 6\}, \{1, 3, 4, 6\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 6\}\}$

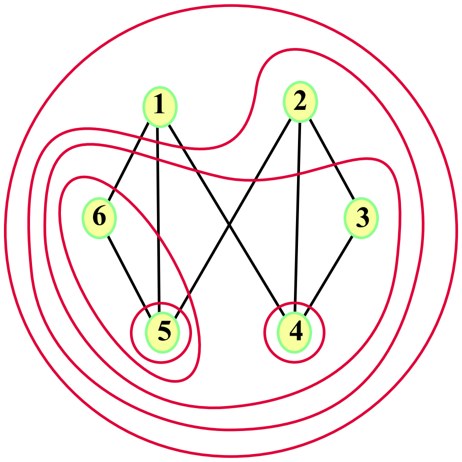
$\mathscr{B}3$



$\mathscr{B}3 = \{ X, \{1\}, \{3\}, \{1, 6\}, \{2, 3\}, \{1, 2, 3, 5, 6\}\}$

$\mathscr{O}3 = \mathscr{B}3 \cup \{\{\}, \{1, 3\}, \{1, 2, 3\}, \{1, 3, 6\}, \{1, 2, 3, 6\}\}$

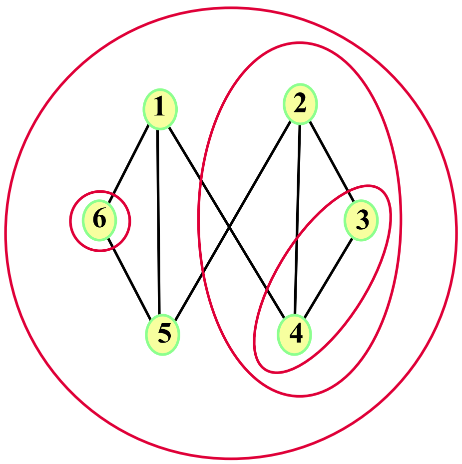
$\mathcal{B}4$



$\mathcal{B}4 = \{ X, \{4\}, \{5\}, \{5, 6\}, \{3, 4, 5, 6\}, \{2, 3, 4, 5, 6\}\}$

$\mathcal{O}4 = \mathcal{B}4 \cup \{\{\}, \{4, 5\}, \{4, 5, 6\}\}$

$\mathcal{B}5$



$\mathcal{B}5 = \{ X, \{6\}, \{3, 4\}, \{2, 3, 4\}\}$

$\mathcal{O}5 = \mathcal{B}5 \cup \{\{\}, \{3, 4, 6\}, \{2, 3, 4, 6\}\}$

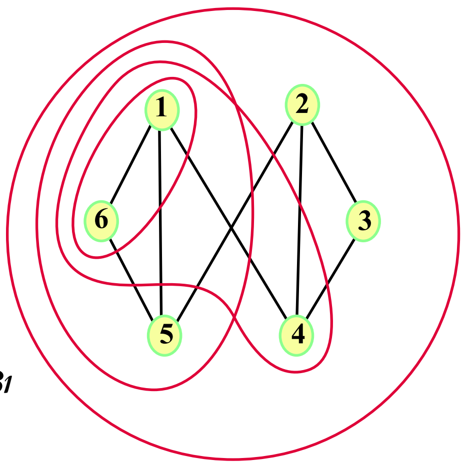
2) For your bases B3 and B4: Either prove that the topological spaces are homeomorphic or that they are not.

They are not homeomorphic.

Proof: Any homeomorphism (bijection and bicontinuous) must take open sets of size 3 to open sets of size 3. That's because the inverse of the homeomorphism is continuous, so f(open) = open, and since f is 1-1, there will be 3 elements in the image of a 3-element set.

Assume there is a homeomorphism from the topology on X with B3 to the topology with B4.
Then $f(\{1,2,3\}) = \{4,5,6\}$.
(The image $\{4,5,6\}$ is the only open set of size three in O4.) However, we also have that $f(\{1,3,6)\} = \{4,5,6\}$ for the same reason.
Therefore, $f(\{1,2,3,6\}) = \{4,5,6\}$ and thus f is not bijective, a contradiction.

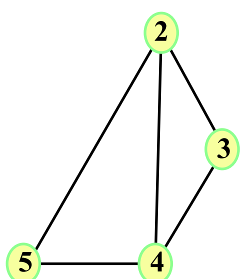
3) The reconnected complement of the graph with respect to a basis element u is found by deleting that set u (and its adjacent edges) and then adding edges to reconnect any two nodes which used to both be attached to that set u with edges. Show that in B1 and B2 there are sets u whose reconnected complement includes edges that are not connected in the original topology. Show that in B5 each set's reconnected complement has edges that are connected in the original topology.



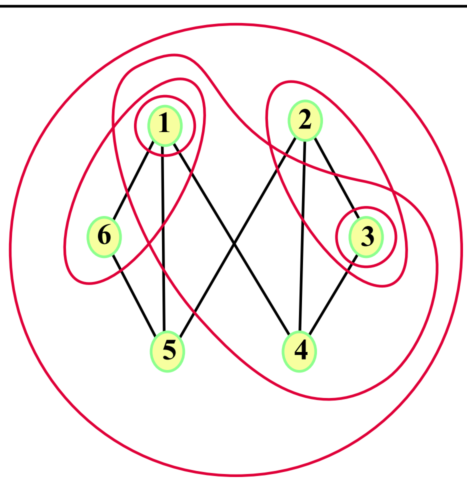
$\mathcal{B}1$

$\mathcal{B}1$ = { X, {1, 6}, {1, 5, 6}, {1, 4, 6}}

We use the basis element u ={1,6}. The reconnected complement is:



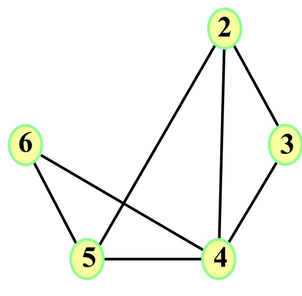
Looking at the basis B1 again, we see that the new edge {4,5} is disconnected as a subspace, since its subspace topology is {{},{4},{5},{4,5}}.



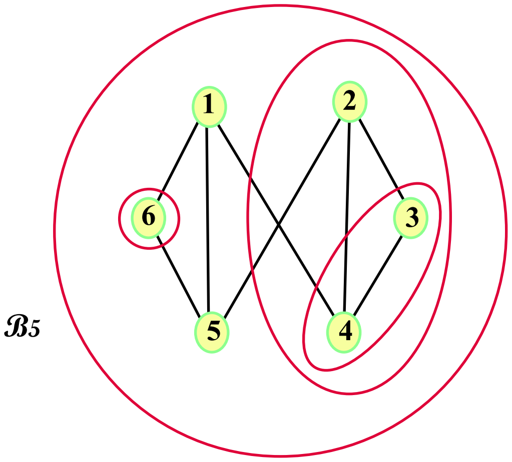
$\mathcal{B}2$

$\mathcal{B}2$ = { X, {1}, {3}, {1, 6}, {2, 3}, {1, 3, 4}}

We use the basis element u ={1}. The reconnected complement is:

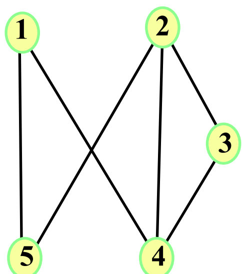


Looking at the basis B2 again, we see that the new edge {4,6} is disconnected as a subspace, since its subspace topology is {{},{4},{6},{4,6}}.

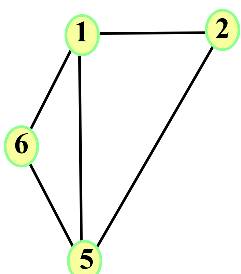


$\mathcal{B}5$ = { X, {6}, {3, 4}, {2, 3, 4}}

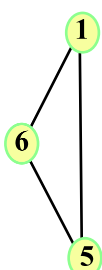
Here is the reconnected complement with respect to {6}. It has no new edges.



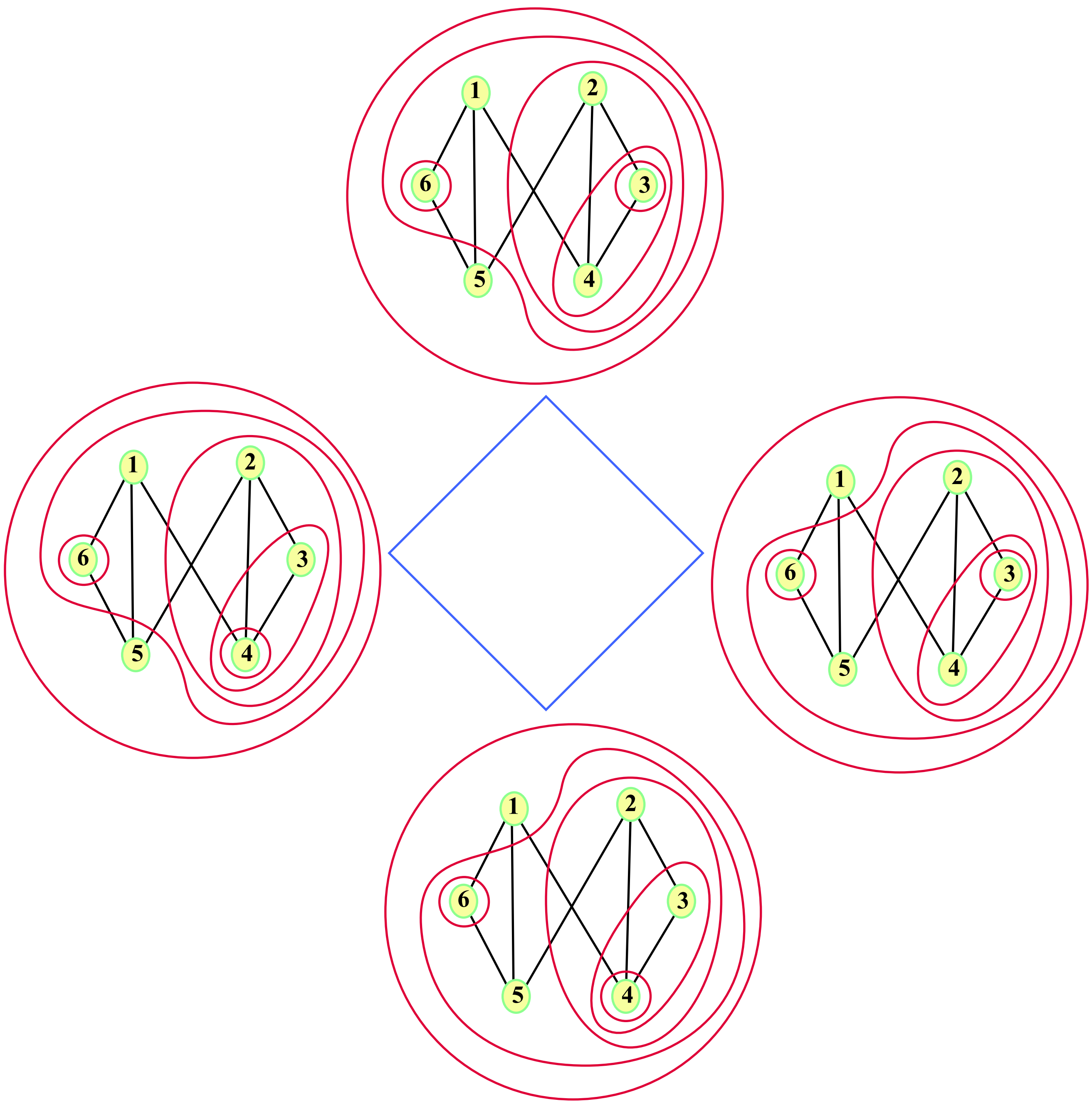
Here is the reconnected complement with respect to {3,4}. It has a new edge {1,2}, which is connected as a subspace.



Here is the reconnected complement with respect to {2,3,4}. It has a no new edges.



4) For B5, find all the ways of adding 2 more sets while preserving the property just checked in #3. To do this just make sure you follow the rules for B3 and B4—no overlaps unless nested. Note that if you find one way, say B6, and remove any set, there is a unique different set to replace it with. Arrange the answers (via names B5a, B5b, etc.) as corners of a polygon with edges of the polygon showing this remove-replace operation.



5) For any two of your bases, find the fundamental group. Either show that those two bases are homotopy equivalent or that they are not.

For B1: $\pi_1(X, O1) = \{e\}$.

For B5: $\pi_1(X, O5) = \{e\}$.

6) For your B1-B5, add an arrow to each edge of the graph, if it can point from inside any basis set to outside that same basis set. There is never an edge pointing both ways, since the edges must be connected as subspaces, so the two nodes will never be inside two separate basis sets.

Check your five results: Do the arrows ever result in a cycle, where following the arrows returns to a previously visited node?________________

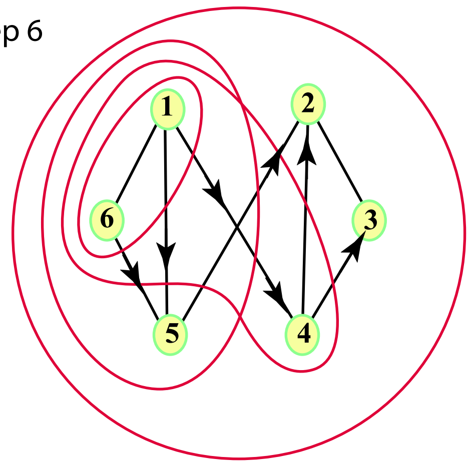
Hint: this is really a double check--there shouldn'y be any cycles of arrows!

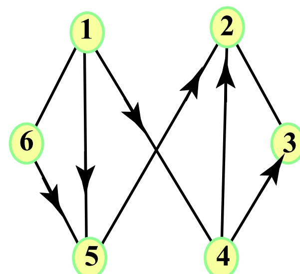
What do you call a field of arrows that never makes a circular path, in Calculus 3?____________

Hint: there are about 5 equivalent right answers here, including terms from physics.

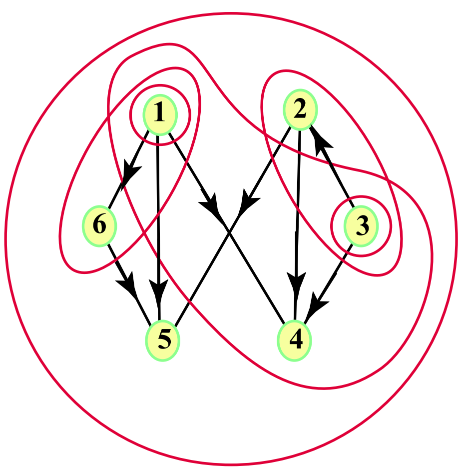
Step 6

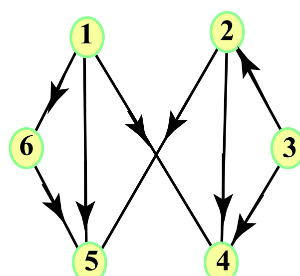
$\mathscr{B}1$



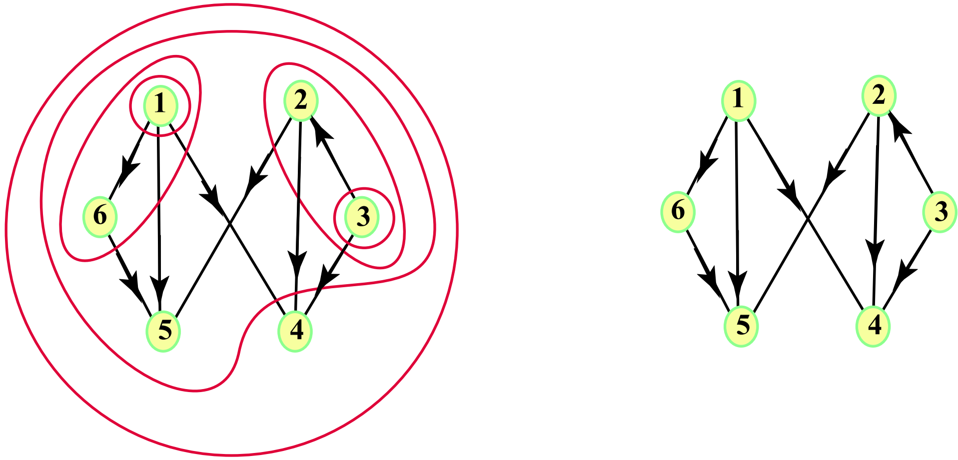


$\mathscr{B}2$

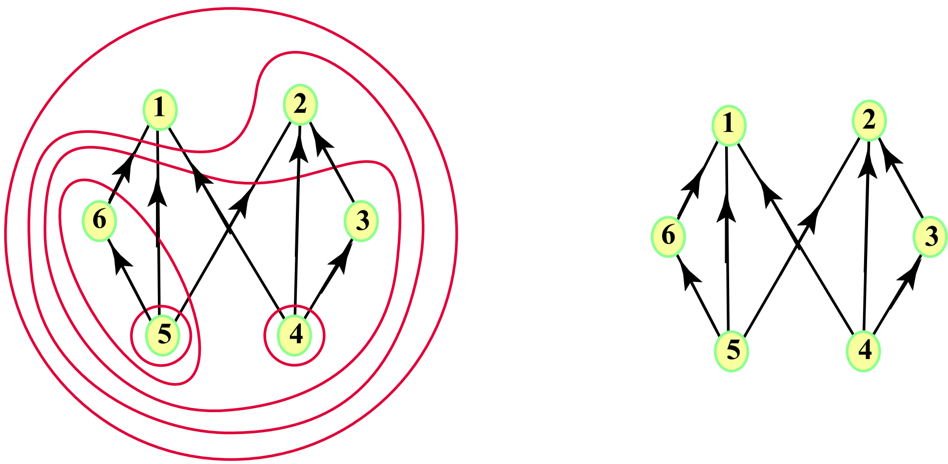




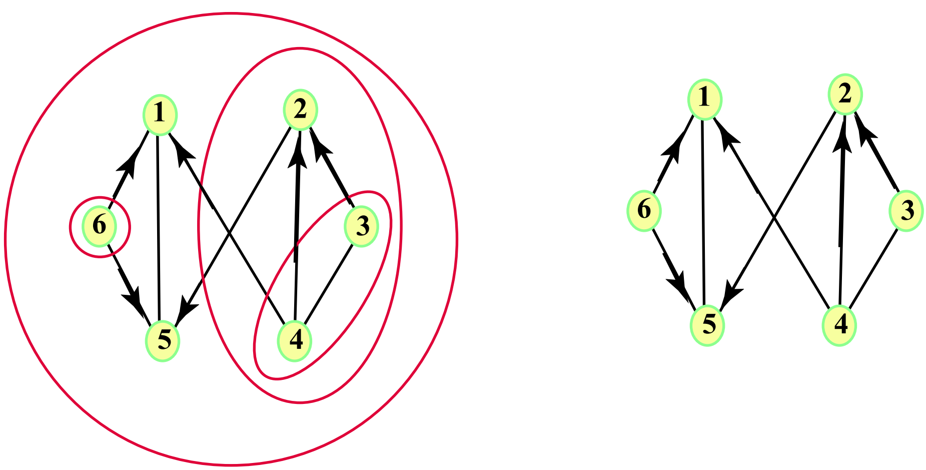
$\mathcal{B}_3$



$\mathcal{B}_4$



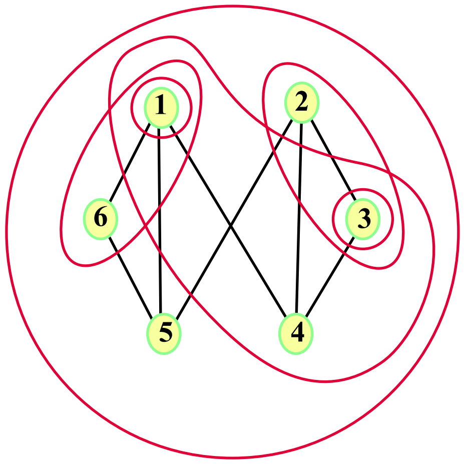
$\mathcal{B}_5$



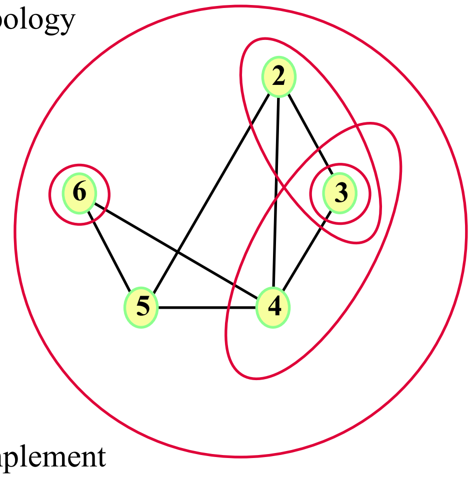
Now, in B2, B3, and B5: choose 1 set (not X), find the reconnected complement with the subspace topology, and try to add arrows to the new graph. When is it possible to add arrows to all the edges?

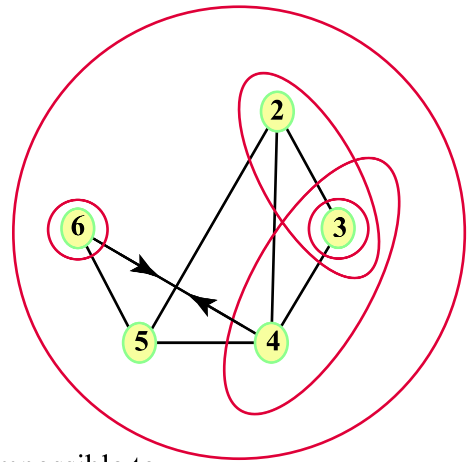
Hint: B3 is the one that will be the "possible to add arrows to all edges" for everyone.

$\mathcal{B}_2$



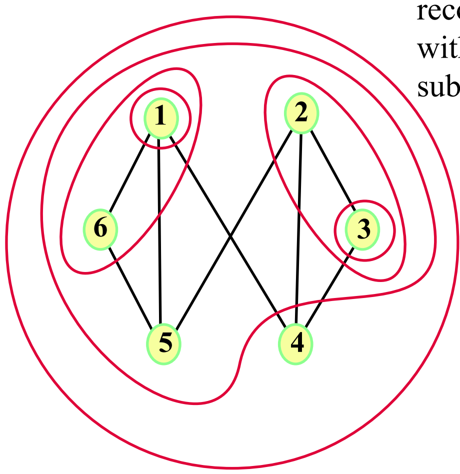
reconnected complement with respect to $\{1\}$, with subspace topology



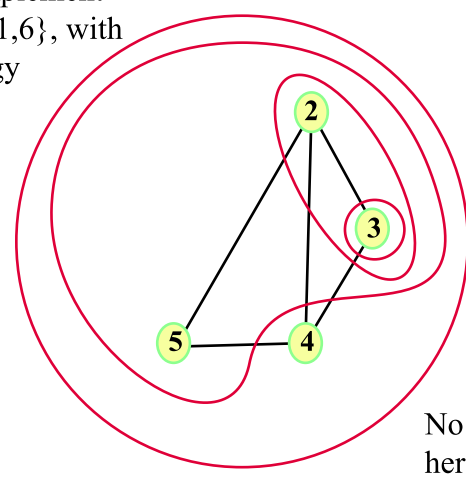


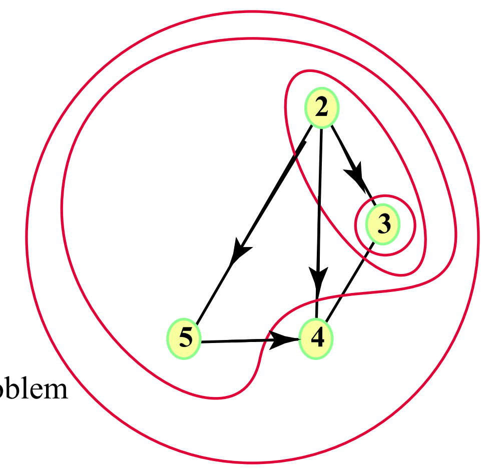
Impossible to orient edge $\{4,6\}$ with the basis sets. (Also edge $\{2,4\}$)

$\mathcal{B}_3$



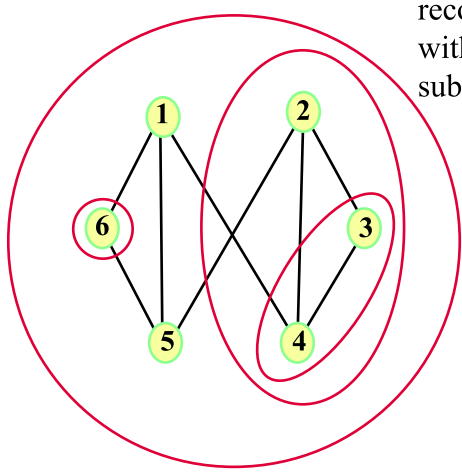
reconnected complement with respect to $\{1,6\}$, with subspace topology



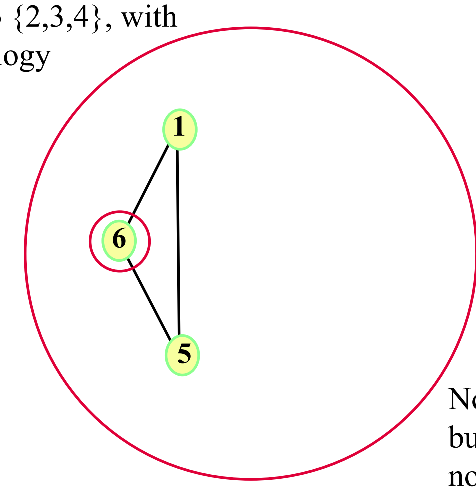


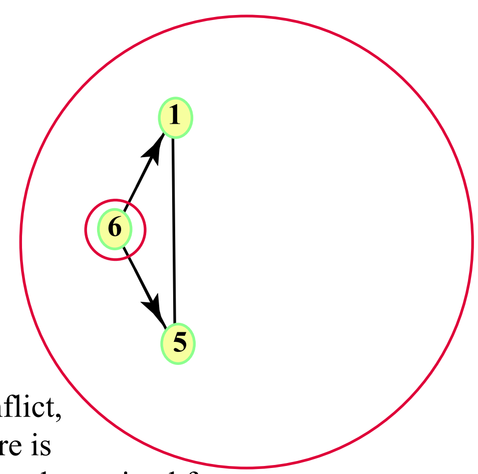
No problem here!

$\mathcal{B}_5$



reconnected complement with respect to $\{2,3,4\}$, with subspace topology





No conflict, but there is no arrow determined for edge $\{1,5\}$.

Choose any one of the following 4 questions. The answers are open ended, but 1-2 pages is about right.

Hint: For the application questions, think about what the arrows might mean, and the advantages of being able to add arrows to reconnected complements. When would deleting a set make sense? When would reconnecting it make sense, in the real world inspired model?