Higher Dimensional Enrichment

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Abstract

Lyubashenko has described enriched 2–categories as categories enriched over $\mathcal{V}$–Cat, the 2–category of categories enriched over a symmetric monoidal $\mathcal{V}$. Here I generalize this to a k–fold monoidal $\mathcal{V}$. The latter is defined as by Balteanu, Fiedorowicz, Schwänzl and Vogt but with the addition of making visible the coherent associators α^i. The symmetric case can easily be recovered. The introduction of this paper proposes a recursive definition of $\mathcal{V}$–n–categories and their morphisms. Then I consider the special case of $\mathcal{V}$–2–categories and give the details of the proof that with their morphisms these form the structure of a 3–category.

1 Introduction

There is an ongoing massive effort to link category theory and geometry, just a part of the broad undertaking known as categorification as described by Baez and Dolan in [Baez and Dolan, 1998]. This effort has as a partial goal that of understanding the categories and functors that correspond to loop spaces and their associated topological functors. Progress towards this goal has been advanced greatly by the recent work of Balteanu, Fiedorowicz, Schwänzl, and Vogt in [Balteanu et.al, 2003] where they show a direct correspondence between k-fold monoidal categories and k-fold loop spaces through the categorical nerve.

As I pursued part of a plan to relate the enrichment functor to topology, I noticed that the concept of higher dimensional enrichment would be important in its relationship to double, triple and further iterations of delooping. The concept of enrichment over a monoidal category is well known, and enriching over the category of categories enriched over a monoidal category is defined, for the case of symmetric categories, in the paper on A_∞–categories by Lyubashenko, [Lyubashenko, 2003]. It seems that it is a good idea to generalize his definition first to the case of an iterated monoidal base category and then to define $\mathcal{V}$–$(n+1)$–categories as categories enriched over $\mathcal{V}$–n–Cat, the $(k-n)$–fold monoidal strict $(n+1)$–category of $\mathcal{V}$–n–categories where $k < n \in \mathbf{N}$. Of course the facts implicit in this last statement must be verified. At each stage of successive enrichments, the number of monoidal products should decrease and the categorical dimension should increase, both by one. This is motivated by topology. When we consider the loop space of a topological space, we see that paths (or

1–cells) in the original are now points (or objects) in the derived space. There is also now automatically a product structure on the points in the derived space, where multiplication is given by concatenation of loops. Delooping is the inverse functor here, and thus involves shifting objects to the status of 1–cells and decreasing the number of ways to multiply.

The concept of a k–fold monoidal strict n–category is easy enough to define as a tensor object in a category of $(k-1)$–fold monoidal n–categories with cartesian product. Thus the products and accompanying associator and interchange transformations are strict bi–n–functors and n–natural transformations respectively. That this sort of structure $((k-n)$–fold monoidal strict $n+1$ category) is possessed by $\mathcal{V}$–n–Cat for $\mathcal{V}$ k–fold monoidal is shown for $n = 1$ and all k in my paper [Forcey, 2003]. A full inductive proof covering all n, k is a work in progress, and this paper fills in one of the gaps; specifically showing how the categorical dimension is increased when the base 2–category is $\mathcal{V}$–Cat. I hope to see eventually how to take the long proof contained here and turn it into part (1) of the induction step in the full proof. Part (2), showing how the number of products is decreasing, should actually be easier since rather than involving new structure at each step it only seems to require a repeating of the axiom checking that is done in the initial case. This is described for the case of $\mathcal{V}$–2–Cat being $(k-2)$–fold monoidal in my previously mentioned paper. In general the decrease is engineered by a shift in index–we define new products $\mathcal{V}$–n–Cat$\times\mathcal{V}$–n–Cat $\to \mathcal{V}$–n–Cat by using cartesian products of object sets and letting hom–objects of the ith product of enriched n–categories be the $(i+1)$th product of hom–objects of the component categories. Symbolically,

$$(\mathcal{A} \otimes_i^{(n)} \mathcal{B})((A, B), (A', B')) = \mathcal{A}(A, A') \otimes_{i+1}^{(n-1)} \mathcal{B}(B, B').$$

The superscript (n) is not necessary since the product is defined by context, but I insert it to make clear at what level of enrichment the product is occurring. Defining the necessary natural transformations for this new product as "based upon" the old ones, and the checking of the axioms that define their structure is briefly mentioned later on in this paper and more fully described in [Forcey, 2003] for certain cases.

The definition of a category enriched over $\mathcal{V}$–n–Cat is simply stated by describing the process as enriching over $\mathcal{V}$–n–Cat with the first of the $k-n$ ordered products. In detail this means that

**Definition 1** *A (small, strict) $\mathcal{V}$–(n+1)–category $\mathcal{U}$ consists of*

1. *A set of objects $|\mathcal{U}|$*

2. *For each pair of objects $A, B \in |\mathcal{U}|$ a $\mathcal{V}$–n–category $\mathcal{U}(A, B)$.*

3. *For each triple of objects $A, B, C \in |\mathcal{U}|$ a $\mathcal{V}$–n–functor*

$$\mathcal{M}_{ABC} : \mathcal{U}(B, C) \otimes_1^{(n)} \mathcal{U}(A, B) \to \mathcal{U}(A, C)$$

2

4. For each object $A \in |\boldsymbol{\mathcal{U}}|$ a $\mathcal{V}$–n–functor

$$\boldsymbol{\mathcal{J}}_A : \boldsymbol{\mathcal{I}}^{(n)} \to \boldsymbol{\mathcal{U}}(A, A)$$

5. Axioms: The $\mathcal{V}$–n–functors that play the role of composition and identity obey commutativity of a pentagonal diagram (associativity axiom) and of two triangular diagrams (unit axioms). This amounts to saying that the functors given by the two legs of each diagram are equal.
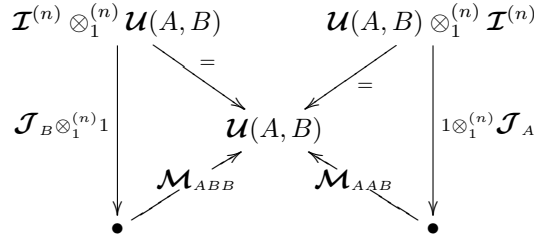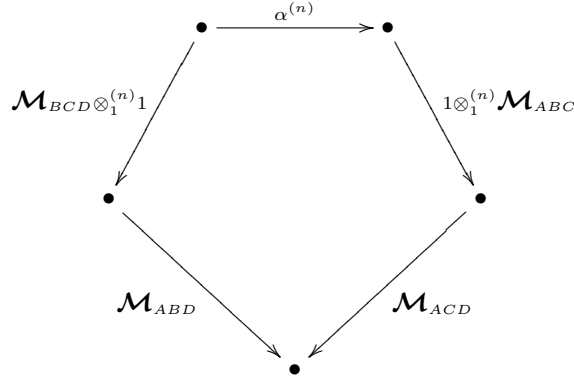
This definition requires that there be a definitions of the unit $\boldsymbol{\mathcal{I}}^{(n)}$ and of $\mathcal{V}$–n–functors in place. Since the axioms talk about equality, the $\mathcal{V}$–n–natural transformations and various higher morphisms are not directly required. They will be utilized however, as soon as the word strict is removed from Definition 1. I will sketch definitions of morphisms in a moment. First, since we hope to realize monoidal structure on $\mathcal{V}$–n–Cat, it needs a recursively defined unit $\mathcal{V}$–n–category.

Definition 2 *The unit object in $\mathcal{V}$–n–Cat is the $\mathcal{V}$–n–category $\boldsymbol{\mathcal{I}}^{(n)}$ with one object $\mathbf{0}$ and with $\boldsymbol{\mathcal{I}}^{(n)}(\mathbf{0}, \mathbf{0}) = \boldsymbol{\mathcal{I}}^{(n-1)}$, where $\boldsymbol{\mathcal{I}}^{(n-1)}$ is the unit object in $\mathcal{V}$–$(n-1)$–Cat. Of course we let $\boldsymbol{\mathcal{I}}^{(0)}$ be I in $\mathcal{V}$. Also $\boldsymbol{\mathcal{M}}_{000} = \boldsymbol{\mathcal{J}}_0 = 1_{\boldsymbol{\mathcal{I}}^{(n)}}$.*

Now we can define the morphisms:

Definition 3 *For two $\mathcal{V}$–n–categories $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{W}}$ a $\mathcal{V}$–n–functor $T : \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{W}}$ is a function on objects $|\boldsymbol{\mathcal{U}}| \to |\boldsymbol{\mathcal{W}}|$ and a family of $\mathcal{V}$–$(n-1)$–functors $T_{UU'} : \boldsymbol{\mathcal{U}}(U, U') \to \boldsymbol{\mathcal{W}}(TU, TU')$. These latter obey commutativity of the usual diagrams.*

3

1. For $U, U', U'' \in |\boldsymbol{\mathcal{U}}|$

$$
\begin{array}{ccc}
\bullet & \xrightarrow{\;\;\boldsymbol{\mathcal{M}}_{UU'U''}\;\;} & \bullet \\
{\scriptstyle T_{U'U''}\otimes_1^{(n-1)}T_{UU'}}\Big\downarrow & & \Big\downarrow{\scriptstyle T_{UU''}} \\
\bullet & \xrightarrow[\;\;\boldsymbol{\mathcal{M}}_{(TU)(TU')(TU'')}\;\;]{} & \bullet
\end{array}
$$

2.

$$
\begin{array}{ccc}
 & & \bullet \\
 & {\scriptstyle \boldsymbol{\mathcal{J}}_{U}}\nearrow & \Big\downarrow{\scriptstyle T_{UU}} \\
\boldsymbol{\mathcal{I}}^{(n-1)} & & \\
 & {\scriptstyle \boldsymbol{\mathcal{J}}_{TU}}\searrow & \bullet
\end{array}
$$

Here a $\mathcal{V}$–0–functor is just a morphism in $\mathcal{V}$.

The 1–cells we have just defined play a special role in the definition of a general k–cell for $k \geq 2$.

Definition 4 *A $\mathcal{V}$–n–k–cell α between $(k-1)$–cells ψ^{k-1} and ϕ^{k-1}, written*

$$
\alpha : \psi^{k-1} \to \phi^{k-1} : \psi^{k-2} \to \phi^{k-2} : ... : \psi^2 \to \phi^2 : F \to G : \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{W}}
$$

where F and G are $\mathcal{V}$–n–functors, is a function sending each $U \in |\boldsymbol{\mathcal{U}}|$ to a $\mathcal{V}$–$((n-k)+1)$–functor

$$
\alpha_U : \boldsymbol{\mathcal{I}}^{((n-k)+1)} \to \boldsymbol{\mathcal{W}}(FU, GU)(\psi_U^2(\mathbf{0}), \phi_U^2(\mathbf{0})...(\psi_U^{k-1}(\mathbf{0}), \phi_U^{k-1}(\mathbf{0}))
$$

in such a way that we have commutativity of the following diagram. Note that the final (curved) equal sign is implied recursively by the diagram for the $(k-1)$–cells.

$$\mathcal{W}(FU',GU')(\psi_{U'}^2(\mathbf{0}),\phi_{U'}^2(\mathbf{0}))...(\psi_{U'}^{k-1}(\mathbf{0}),\phi_{U'}^{k-1}(\mathbf{0}))$$
$$\otimes_{k-1}^{((n-k)+1)}\mathcal{W}(FU,FU')(F(x_2),F(y_2))...(F(x_{k-1}),F(y_{k-1}))$$

$$\alpha_{U'}\otimes_{k-1}^{((n-k)+1)}F$$

$$\mathcal{M}$$

$$\mathcal{I}^{((n-k)+1)}\otimes_{k-1}^{((n-k)+1)}\mathcal{U}(U,U')(x_2,y_2)...(x_{k-1},y_{k-1})$$

$$\mathcal{W}(FU,GU')(\psi_{U'}^2(\mathbf{0})F(x_2),\phi_{U'}^2(\mathbf{0})F(y_2))...(\psi_{U'}^{k-1}(\mathbf{0})F(x_{k-1}),\phi_{U'}^{k-1}(\mathbf{0})F(y_{k-1}))$$

$$=$$

$$\mathcal{U}(U,U')(x_2,y_2)...(x_{k-1},y_{k-1})$$

$$=$$

$$\mathcal{U}(U,U')(x_2,y_2)...(x_{k-1},y_{k-1})\otimes_{k-1}^{((n-k)+1)}\mathcal{I}^{((n-k)+1)}$$

$$\mathcal{W}(FU,GU')(G(x_2)\psi_U^2(\mathbf{0}),G(y_2)\phi_U^2(\mathbf{0}))...(G(x_{k-1})\psi_U^{k-1}(\mathbf{0}),G(y_{k-1})\phi_U^{k-1}(\mathbf{0}))$$

$$G\otimes_{k-1}^{((n-k)+1)}\alpha_U$$

$$\mathcal{M}$$

$$\mathcal{W}(GU,GU')(G(x_2),G(y_2))...(G(x_{k-1}),G(y_{k-1}))$$
$$\otimes_{k-1}^{((n-k)+1)}\mathcal{W}(FU,GU)(\psi_U^2(\mathbf{0}),\phi_U^2(\mathbf{0})...(\psi_U^{k-1}(\mathbf{0}),\phi_U^{k-1}(\mathbf{0}))$$

Thus for a given value of n there are k–cells up to $k = n + 1$, making $\mathcal{V}$–n–Cat a potential $(n+1)$–category. This last definition is best grasped by looking at examples. The cases for $n = 1, 2$ are given in detail in the following section.

2 Review of Definitions

In this section I briefly review the definitions of a category enriched over a monoidal category $\mathcal{V}$, a category enriched over an iterated monoidal category, and an enriched 2–category. I begin with the basic definitions of enrichment, included due to how often they are referred to and followed as models in the rest of the paper. This first set of definitions can be found with more detail in [Kelly, 1982] and [Eilenberg and Kelly, 1965].

Definition 5 *For our purposes a monoidal category is a category $\mathcal{V}$ together with a functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and an object I such that*

1. *$\otimes$ is associative up to the coherent natural transformations α. The coherence axiom is given by the commuting pentagon*

2. *I is a strict 2-sided unit for $\otimes$.*

Definition 6 *A (small) $\mathcal{V}$ –Category $\mathcal{A}$ is a set $|\mathcal{A}|$ of objects, a hom-object $\mathcal{A}(A, B) \in |\mathcal{V}|$ for each pair of objects of $\mathcal{A}$, a family of composition morphisms $M_{ABC} : \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \to \mathcal{A}(A, C)$ for each triple of objects, and an identity element $j_A : I \to \mathcal{A}(A, A)$ for each object. The composition morphisms are subject to the associativity axiom which states that the following pentagon commutes*

and to the unit axioms which state that both the triangles in the following diagram commute

$$
\begin{array}{ccc}
I \otimes \mathcal{A}(A,B) & & \mathcal{A}(A,B) \otimes I \\
& \searrow^{=} \quad \swarrow^{=} & \\
j_B \otimes 1 \downarrow \quad \mathcal{A}(A,B) \quad \downarrow 1 \otimes j_A & & \\
& \nearrow^{M_{ABB}} \quad \nwarrow^{M_{AAB}} & \\
\mathcal{A}(B,B) \otimes \mathcal{A}(A,B) & & \mathcal{A}(A,B) \otimes \mathcal{A}(A,A)
\end{array}
$$

Definition 7 *For $\mathcal{V}$–categories $\mathcal{A}$ and $\mathcal{B}$, a $\mathcal{V}$–functor $T : \mathcal{A} \to \mathcal{B}$ is a function $T : |\mathcal{A}| \to |\mathcal{B}|$ and a family of $hom-object$ morphisms $T_{AB} : \mathcal{A}(A,B) \to \mathcal{B}(TA,TB)$ in $\mathcal{V}$ indexed by pairs $A, B \in |\mathcal{A}|$. The usual rules for a functor that state $T(f \circ g) = Tf \circ Tg$ and $T1_A = 1_{TA}$ become in the enriched setting, respectively, the commuting diagrams*

$$
\begin{array}{ccc}
\mathcal{A}(B,C) \otimes \mathcal{A}(A,B) & \xrightarrow{\quad M \quad} & \mathcal{A}(A,C) \\
\downarrow^{T \otimes T} & & \downarrow^{T} \\
\mathcal{B}(TB,TC) \otimes \mathcal{B}(TA,TB) & \xrightarrow{\quad M \quad} & \mathcal{B}(TA,TC)
\end{array}
$$

and

$$
\begin{array}{ccc}
& & \mathcal{A}(A,A) \\
& \nearrow^{j_A} & \downarrow \\
I & & \downarrow^{T_{AA}} \\
& \searrow_{j_{TA}} & \downarrow \\
& & \mathcal{B}(TA,TA).
\end{array}
$$

$\mathcal{V}$–functors can be composed to form a category called $\mathcal{V}$–Cat. This category is actually enriched over **Cat***, the category of (small) categories with cartesian product.*

Definition 8 *For $\mathcal{V}$–functors $T, S : \mathcal{A} \to \mathcal{B}$ a $\mathcal{V}$–natural transformation $\alpha : T \to S : \mathcal{A} \to \mathcal{B}$ is an $|\mathcal{A}|$–indexed family of morphisms $\alpha_A : I \to \mathcal{B}(TA, SA)$ satisfying the $\mathcal{V}$–naturality condition expressed by the commutativity of*

$$
\begin{array}{ccc}
& I \otimes \mathcal{A}(A,B) \xrightarrow{\ \alpha_B \otimes T_{AB}\ } \mathcal{B}(TB,SB) \otimes \mathcal{B}(TA,TB) & \\
{}^{=}\nearrow & & \searrow {}^{M} \\
\mathcal{A}(A,B) & & \mathcal{B}(TA,SB) \\
{}^{=}\searrow & & \nearrow {}^{M} \\
& \mathcal{A}(A,B) \otimes I \xrightarrow[\ S_{AB} \otimes \alpha_A\ ]{} \mathcal{B}(SA,SB) \otimes \mathcal{B}(TA,SA) &
\end{array}
$$

For two $\mathcal{V}$–functors T, S to be equal is to say $TA = SA$ for all A and for the $\mathcal{V}$–natural isomorphism α between them to have components $\alpha_A = j_{TA}$. This latter implies equality of the hom–object morphisms: $T_{AB} = S_{AB}$ for all pairs of objects. The implication is seen by combining the second diagram in Definition 6 with all the diagrams in Definitions 7 and 8.

The fact that $\mathcal{V}$–Cat has the structure of a 2–category is demonstrated in [Kelly, 1982]. Now we review the transfer to enriching over a k–fold monoidal category. The latter sort of category was developed and defined in [Balteanu et.al, 2003]. The authors describe its structure as arising from its description as a monoid in the category of $(k-1)$–fold monoidal categories. Here is that definition altered only slightly to make visible the coherent associators as in [Forcey, 2003]. In that paper I describe its structure as arising from its description as a tensor object in the category of $(k-1)$–fold monoidal categories

Definition 9 *An n-fold monoidal category is a category $\mathcal{V}$ with the following structure.*

1. *There are n distinct multiplications*

$$
\otimes_1, \otimes_2, \ldots, \otimes_n : \mathcal{V} \times \mathcal{V} \to \mathcal{V}
$$

for each of which the associativity pentagon commutes

$$
\begin{array}{ccc}
((U \otimes_i V) \otimes_i W) \otimes_i X & \xrightarrow{\ \alpha^i_{UVW} \otimes_i 1_X\ } & (U \otimes_i (V \otimes_i W)) \otimes_i X \\
{\scriptstyle \alpha^i_{(U \otimes_i V)WX}}\swarrow & & \searrow {\scriptstyle \alpha^i_{U(V \otimes_i W)X}} \\
(U \otimes_i V) \otimes_i (W \otimes_i X) & & U \otimes_i ((V \otimes_i W) \otimes_i X) \\
{\scriptstyle \alpha^i_{UV(W \otimes_i X)}}\searrow & & \swarrow {\scriptstyle 1_U \otimes_i \alpha^i_{VWX}} \\
& U \otimes_i (V \otimes_i (W \otimes_i X)) &
\end{array}
$$

$\mathcal{V}$ has an object I which is a strict unit for all the multiplications.

2. For each pair (i,j) such that $1 \le i < j \le n$ there is a natural transformation

$$\eta^{ij}_{ABCD} : (A \otimes_j B) \otimes_i (C \otimes_j D) \to (A \otimes_i C) \otimes_j (B \otimes_i D).$$

These natural transformations η^{ij} are subject to the following conditions:

(a) Internal unit condition: $\eta^{ij}_{ABII} = \eta^{ij}_{IIAB} = 1_{A \otimes_j B}$

(b) External unit condition: $\eta^{ij}_{AIBI} = \eta^{ij}_{IAIB} = 1_{A \otimes_i B}$

(c) Internal associativity condition: The following diagram commutes

$$
\begin{array}{ccc}
((U \otimes_j V) \otimes_i (W \otimes_j X)) \otimes_i (Y \otimes_j Z) & \xrightarrow{\eta^{ij}_{UVWX} \otimes_i 1_{Y \otimes_j Z}} & ((U \otimes_i W) \otimes_j (V \otimes_i X)) \otimes_i (Y \otimes_j Z) \\
\downarrow{\alpha^i} & & \downarrow{\eta^{ij}_{(U \otimes_i W)(V \otimes_i X)YZ}} \\
(U \otimes_j V) \otimes_i ((W \otimes_j X) \otimes_i (Y \otimes_j Z)) & & ((U \otimes_i W) \otimes_i Y) \otimes_j ((V \otimes_i X) \otimes_i Z) \\
\downarrow{1_{U \otimes_j V} \otimes_i \eta^{ij}_{WXYZ}} & & \downarrow{\alpha^i \otimes_j \alpha^i} \\
(U \otimes_j V) \otimes_i ((W \otimes_i Y) \otimes_j (X \otimes_i Z)) & \xrightarrow{\eta^{ij}_{UV(W \otimes_i Y)(X \otimes_i Z)}} & (U \otimes_i (W \otimes_i Y)) \otimes_j (V \otimes_i (X \otimes_i Z))
\end{array}
$$

(d) External associativity condition: The following diagram commutes

$$
\begin{array}{ccc}
((U \otimes_j V) \otimes_j W) \otimes_i ((X \otimes_j Y) \otimes_j Z) & \xrightarrow{\eta^{ij}_{(U \otimes_j V)W(X \otimes_j Y)Z}} & ((U \otimes_j V) \otimes_i (X \otimes_j Y)) \otimes_j (W \otimes_i Z) \\
\downarrow{\alpha^j \otimes_i \alpha^j} & & \downarrow{\eta^{ij}_{UVXY} \otimes_j 1_{W \otimes_i Z}} \\
(U \otimes_j (V \otimes_j W)) \otimes_i (X \otimes_j (Y \otimes_j Z)) & & ((U \otimes_i X) \otimes_j (V \otimes_i Y)) \otimes_j (W \otimes_i Z) \\
\downarrow{\eta^{ij}_{U(V \otimes_j W)X(Y \otimes_j Z)}} & & \downarrow{\alpha^j} \\
(U \otimes_i X) \otimes_j ((V \otimes_j W) \otimes_i (Y \otimes_j Z)) & \xrightarrow{1_{U \otimes_i X} \otimes_j \eta^{ij}_{VWYZ}} & (U \otimes_i X) \otimes_j ((V \otimes_i Y) \otimes_j (W \otimes_i Z))
\end{array}
$$

(e) Finally it is required that for each triple (i,j,k) satisfying $1 \le i < j < k \le n$ the giant hexagonal interchange diagram commutes.

$$((A \otimes_k A') \otimes_j (B \otimes_k B')) \otimes_i ((C \otimes_k C') \otimes_j (D \otimes_k D'))$$

$$\eta^{jk}_{AA'BB'} \otimes_i \eta^{jk}_{CC'DD'} \qquad\qquad \eta^{ij}_{(A \otimes_k A')(B \otimes_k B')(C \otimes_k C')(D \otimes_k D')}$$

$$((A \otimes_j B) \otimes_k (A' \otimes_j B')) \otimes_i ((C \otimes_j D) \otimes_k (C' \otimes_j D')) \qquad ((A \otimes_k A') \otimes_i (C \otimes_k C')) \otimes_j ((B \otimes_k B') \otimes_i (D \otimes_k D'))$$

$$\eta^{ik}_{(A \otimes_j B)(A' \otimes_j B')(C \otimes_j D)(C' \otimes_j D')} \qquad\qquad \eta^{ik}_{AA'CC'} \otimes_j \eta^{ik}_{BB'DD'}$$

$$((A \otimes_j B) \otimes_i (C \otimes_j D)) \otimes_k ((A' \otimes_j B') \otimes_i (C' \otimes_j D')) \qquad ((A \otimes_i C) \otimes_k (A' \otimes_i C')) \otimes_j ((B \otimes_i D) \otimes_k (B' \otimes_i D'))$$

$$\eta^{ij}_{ABCD} \otimes_k \eta^{ij}_{A'B'C'D'} \qquad\qquad \eta^{jk}_{(A \otimes_i C)(A' \otimes_i C')(B \otimes_i D)(B' \otimes_i D')}$$

$$((A \otimes_i C) \otimes_j (B \otimes_i D)) \otimes_k ((A' \otimes_i C') \otimes_j (B' \otimes_i D'))$$

The authors of [Balteanu et.al, 2003] remark that a symmetric monoidal category is n-fold monoidal for all n. This they demonstrate by letting

$$\otimes_1 = \otimes_2 = \ldots = \otimes_n = \otimes$$

and defining (associators added by myself)

$$\eta^{ij}_{ABCD} = \alpha^{-1} \circ (1_A \otimes \alpha) \circ (1_A \otimes (c_{BC} \otimes 1_D)) \circ (1_A \otimes \alpha^{-1}) \circ \alpha$$

for all $i < j$. Here $c_{BC} : B \otimes C \to C \otimes B$ is the symmetry natural transformation. This provides the hint that enriching over a k–fold monoidal category is precisely a generalization of enriching over a symmetric category. In the symmetric case, to define a product in $\mathcal{V}$–Cat, we need c_{BC} in order to create a middle exchange morphism m. To describe products in $\mathcal{V}$–Cat for $\mathcal{V}$ k–fold monoidal we simply use $m = \eta$.

Categories enriched over k–fold monoidal $\mathcal{V}$ are carefully defined in [Forcey, 2003], where they are shown to be the objects of a $(k-1)$–fold monoidal 2–category. Here we need only the definitions. Simply put, a category enriched over a k–fold monoidal $\mathcal{V}$ is a category enriched in the usual sense over $(\mathcal{V}, \otimes_1, I, \alpha)$. The other $k-1$ products in $\mathcal{V}$ are used up in the structure of $\mathcal{V}$–Cat. I will always denote the product(s) in $\mathcal{V}$–Cat with a superscript in parentheses that corresponds to the level of enrichment of the components of their domain. The product(s) in $\mathcal{V}$ should logically then have a superscript (0) but I have suppressed this for brevity and to agree with my sources. For $\mathcal{V}$ k–fold monoidal we define the ith product of $\mathcal{V}$–categories $\mathcal{A} \otimes_i^{(1)} \mathcal{B}$ to have objects $\in |\mathcal{A}| \times |\mathcal{B}|$ and to have hom–objects $\in |\mathcal{V}|$ given by

$$(\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A, B), (A', B')) = \mathcal{A}(A, A') \otimes_{i+1} \mathcal{B}(B, B').$$

Immediately we see that $\mathcal{V}$–Cat is $(k-1)$–fold monoidal by definition. (The full proof of this is in [Forcey, 2003].) The composition morphisms are

$$M_{(A,B)(A',B')(A'',B'')} : (\mathcal{A}\otimes_i^{(1)}\mathcal{B})((A',B'),(A'',B''))\otimes_1(\mathcal{A}\otimes_i^{(1)}\mathcal{B})((A,B),(A',B')) \to (\mathcal{A}\otimes_i^{(1)}\mathcal{B})((A,B),(A'',B''))$$

given by

$$(\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A',B'),(A'',B'')) \otimes_1 (\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A,B),(A',B'))$$

$$\|$$

$$(\mathcal{A}(A',A'') \otimes_{i+1} \mathcal{B}(B',B'')) \otimes_1 (\mathcal{A}(A,A') \otimes_{i+1} \mathcal{B}(B,B'))$$

$$\eta^{1,i+1}\Big\downarrow$$

$$(\mathcal{A}(A',A'') \otimes_1 \mathcal{A}(A,A')) \otimes_{i+1} (\mathcal{B}(B',B'') \otimes_1 \mathcal{B}(B,B'))$$

$$M_{AA'A''}\otimes_2 M_{BB'B''}\Big\downarrow$$

$$(\mathcal{A}(A,A'') \otimes_{i+1} \mathcal{B}(B,B''))$$

$$\|$$

$$(\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A,B),(A'',B''))$$

The identity element is given by $j_{(A,B)} =$

$$I = I \otimes_{i+1} I$$

$$j_A\otimes_{i+1}j_B\Big\downarrow$$

$$\mathcal{A}(A,A) \otimes_{i+1} \mathcal{B}(B,B)$$

$$\|$$

$$(\mathcal{A} \otimes_i^{(1)} \mathcal{B})((A,B),(A,B))$$

The unit object in $\mathcal{V}$–1–Cat $= \mathcal{V}$–Cat is the enriched category $\mathcal{I}^{(1)} = \mathcal{I}$ where $|\mathcal{I}| = \{0\}$ and $\mathcal{I}(0,0) = I$. Of course $M_{000} = 1_I = j_0$.

That each product $\otimes_i^{(1)}$ thus defined is a 2–bi–functor $\mathcal{V}$–Cat $\times$ $\mathcal{V}$–Cat $\to \mathcal{V}$–Cat is seen easily. Its action on enriched functors and natural transformations is to form formal products using $\otimes_{i+1}$ of their associated morphisms. That the result is a valid enriched functor or natural transformation always follows from the naturality of η.

Associativity in $\mathcal{V}$–Cat must hold for each $\otimes_i^{(1)}$. The components of the 2–natural isomorphism $\alpha^{(1)i}$

$$\alpha_{\mathcal{ABC}}^{(1)i} : (\mathcal{A} \otimes_i^{(1)} \mathcal{B}) \otimes_i^{(1)} \mathcal{C} \to \mathcal{A} \otimes_i^{(1)} (\mathcal{B} \otimes_i^{(1)} \mathcal{C})$$

are $\mathcal{V}$–functors that send ((A,B),C) to (A,(B,C)) and whose hom-components

$$\alpha_{\mathcal{ABC}_{((A,B),C)((A',B'),C')}}^{(1)i} : [(\mathcal{A}\otimes_i^{(1)}\mathcal{B})\otimes_i^{(1)}\mathcal{C}](((A,B),C),((A',B'),C')) \to [\mathcal{A}\otimes_i^{(1)}(\mathcal{B}\otimes_i^{(1)}\mathcal{C})]((A,(B,C)),(A',(B',C')))$$

are given by

$$\alpha^{(1)i}_{\mathcal{ABC}_{((A,B),C)((A',B'),C')}} = \alpha^{i+1}_{\mathcal{A}(A,A')\mathcal{B}(B,B')\mathcal{C}(C,C')}.$$

Now for the interchange 2–natural transformations $\eta^{(1)ij}$ for $j \geq i+1$. We define the component morphisms $\eta^{(1)i,j}_{\mathcal{ABCD}}$ that make a 2–natural transformation between 2–functors. Each component must be an enriched functor. Their action on objects is to send $((A,B),(C,D)) \in \left|(\mathcal{A} \otimes^{(1)}_j \mathcal{B}) \otimes^{(1)}_i (\mathcal{C} \otimes^{(1)}_j \mathcal{D})\right|$ to $((A,C),(B,D)) \in \left|(\mathcal{A} \otimes^{(1)}_i \mathcal{C}) \otimes^{(1)}_j (\mathcal{B} \otimes^{(1)}_i \mathcal{D})\right|$. The hom–object morphisms are given by

$$\eta^{(1)i,j}_{\mathcal{ABCD}_{(ABCD)(A'B'C'D')}} = \eta^{i+1,j+1}_{\mathcal{A}(A,A')\mathcal{B}(B,B')\mathcal{C}(C,C')\mathcal{D}(D,D')}.$$

That the axioms regarding the associators and interchange transformations are all obeyed is established in [Forcey, 2003].

We now define categories enriched over $\mathcal{V}$–Cat. These are defined for the symmetric case in [Lyubashenko, 2003]. Here as in [Forcey, 2003] the definition of $\mathcal{V}$–2–category is generalized for $\mathcal{V}$ a k–fold monoidal category with $k \geq 2$. The definition for symmetric monoidal $\mathcal{V}$ can be easily recovered just by letting $\otimes_1 = \otimes_2 = \otimes$, $\alpha^2 = \alpha^1 = \alpha$ and $\eta = m$.

Definition 10 *A (small, strict) $\mathcal{V}$–2–category $\mathcal{U}$ consists of*

1. *A set of objects $|\mathcal{U}|$*

2. *For each pair of objects $A, B \in |\mathcal{U}|$ a $\mathcal{V}$–category $\mathcal{U}(A,B)$.*

   Of course then $\mathcal{U}(A,B)$ consists of a set of objects (which play the role of the 1–cells in a 2–category) and for each pair $f, g \in |\mathcal{U}(A,B)|$ an object $\mathcal{U}(A,B)(f,g) \in \mathcal{V}$ (which plays the role of the hom–set of 2–cells in a 2–category.) Thus the vertical composition morphisms of these hom_2–objects are in $\mathcal{V}$:

   $$M_{fgh} : \mathcal{U}(A,B)(g,h) \otimes_1 \mathcal{U}(A,B)(f,g) \to \mathcal{U}(A,B)(f,h)$$

   Also, the vertical identity for a 1-cell object $a \in |\mathcal{U}(A,B)|$ is $j_a : I \to \mathcal{U}(A,B)(a,a)$. The associativity and the units of vertical composition are then those given by the respective axioms of enriched categories.

3. *For each triple of objects $A, B, C \in |\mathcal{U}|$ a $\mathcal{V}$–functor*

   $$\mathcal{M}_{ABC} : \mathcal{U}(B,C) \otimes^{(1)}_1 \mathcal{U}(A,B) \to \mathcal{U}(A,C)$$

   Often I repress the subscripts. We denote $\mathcal{M}(h,f)$ as hf.

   The family of morphisms indexed by pairs of objects $(g,f),(g',f') \in \left|\mathcal{U}(B,C) \otimes^{(1)}_1 \mathcal{U}(A,B)\right|$ furnishes the direct analogue of horizontal composition of 2-cells as can be seen by observing their domain and range in $\mathcal{V}$:

   $$\mathcal{M}_{ABC_{(g,f)(g',f')}} : [\mathcal{U}(B,C) \otimes^{(1)}_1 \mathcal{U}(A,B)]((g,f),(g',f')) \to \mathcal{U}(A,C)(gf,g'f')$$

Recall that

$$[\boldsymbol{\mathcal{U}}(B,C) \otimes_1^{(1)} \boldsymbol{\mathcal{U}}(A,B)]((g,f),(g',f')) = \boldsymbol{\mathcal{U}}(B,C)(g,g') \otimes_2 \boldsymbol{\mathcal{U}}(A,B)(f,f').$$

4. *For each object $A \in |\boldsymbol{\mathcal{U}}|$ a $\mathcal{V}$–functor*

$$\mathcal{J}_A : \mathcal{I} \to \boldsymbol{\mathcal{U}}(A,A)$$

We denote $\mathcal{J}_A(0)$ as 1_A.

5. *(Associativity axiom of a strict $\mathcal{V}$–2–category.) We require a commuting pentagon. Since the morphisms are $\mathcal{V}$–functors this amounts to saying that the functors given by the two legs of the diagram are equal. For objects we have the equality $(fg)h = f(gh)$.*

For the hom–object morphisms we have the following family of commuting diagrams for associativity, where the first bullet represents

$$[(\boldsymbol{\mathcal{U}}(C,D) \otimes_1^{(1)} \boldsymbol{\mathcal{U}}(B,C)) \otimes_1^{(1)} \boldsymbol{\mathcal{U}}(A,B)](((f,g),h),((f',g'),h'))$$

and the reader may fill in the others
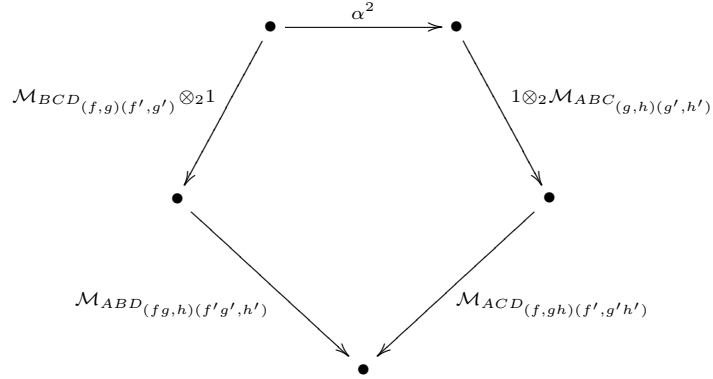
The heuristic diagram for this commutativity is
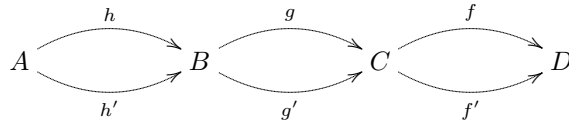
6. *(Unit axioms of a strict $\mathcal{V}$–2–category.) We require commuting triangles. For objects we have the equality $f1_A = f = 1_B f$. For the unit morphisms we have that the triangles in the following diagram commute.*

$$[\mathcal{I} \otimes_1^{(1)} \boldsymbol{\mathcal{U}}(A,B)]((0,f),(0,g)) \qquad\qquad [\boldsymbol{\mathcal{U}}(A,B) \otimes_1^{(1)} \mathcal{I}]((f,0),(g,0))$$

$$\boldsymbol{\mathcal{U}}(A,B)(f,g)$$

with maps $\mathcal{J}_{B_{00}} \otimes_2 1$, $=$, $1 \otimes_2 \mathcal{J}_{A_{00}}$, $\mathcal{M}_{ABB_{(1_B,f)(1_B,g)}}$, $\mathcal{M}_{AAB_{(f,1_A)(g,1_A)}}$

$$[\boldsymbol{\mathcal{U}}(B,B) \otimes_1^{(1)} \boldsymbol{\mathcal{U}}(A,B)]((1_B,f),(1_B,g)) \qquad [\boldsymbol{\mathcal{U}}(A,B) \otimes_1^{(1)} \boldsymbol{\mathcal{U}}(A,A)]((f,1_A),(g,1_A))$$

The heuristic diagrams for this commutativity are

Consequences of $\mathcal{V}$–functoriality of $\mathcal{M}$ and $\mathcal{J}$: First the $\mathcal{V}$–functoriality of $\mathcal{M}$ implies that the following (expanded) diagram commutes

$$(\boldsymbol{\mathcal{U}}(B,C)(k,m) \otimes_1 \boldsymbol{\mathcal{U}}(B,C)(h,k)) \otimes_2 (\boldsymbol{\mathcal{U}}(A,B)(g,l) \otimes_1 \boldsymbol{\mathcal{U}}(A,B)(f,g))$$

with maps $\eta^{1,2}$, $M_{hkm} \otimes_2 M_{fgl}$

$$(\boldsymbol{\mathcal{U}}(B,C)(k,m) \otimes_2 \boldsymbol{\mathcal{U}}(A,B)(g,l)) \otimes_1 (\boldsymbol{\mathcal{U}}(B,C)(h,k) \otimes_2 \boldsymbol{\mathcal{U}}(A,B)(f,g)) \qquad \boldsymbol{\mathcal{U}}(B,C)(h,m) \otimes_2 \boldsymbol{\mathcal{U}}(A,B)(f,l)$$

with maps $\mathcal{M}_{ABC_{(k,g)(m,l)}} \otimes_1 \mathcal{M}_{ABC_{(h,f)(k,g)}}$, $\mathcal{M}_{ABC_{(h,f)(m,l)}}$

$$\boldsymbol{\mathcal{U}}(A,C)(kg,ml) \otimes_1 \boldsymbol{\mathcal{U}}(A,C)(hf,kg) \xrightarrow{\ M_{(hf)(kg)(ml)}\ } \boldsymbol{\mathcal{U}}(A,C)(hf,ml)$$

The heuristic diagram is

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Secondly the $\mathcal{V}$–functoriality of $\mathcal{M}$ implies that the following (expanded) diagram commutes

$$\mathcal{U}(B,C)(g,g) \otimes_2 \mathcal{U}(A,B)(f,f)$$

$$I \xrightarrow{j_g \otimes_2 j_f}$$

$$\downarrow \mathcal{M}_{ABC_{(g,f)(g,f)}}$$

$$I \xrightarrow{j_{gf}} \mathcal{U}(A,C)(gf,gf)$$

The heuristic diagram here is

In addition, the $\mathcal{V}$–functoriality of $\mathcal{J}$ implies that the following (expanded) diagram commutes

$$\mathcal{I}(0,0)$$

$$I \xrightarrow{j_0}$$

$$\downarrow \mathcal{J}_{A_{00}}$$

$$I \xrightarrow{j_{1_A}} \mathcal{U}(A,A)(1_A,1_A)$$

Which means that

$$\mathcal{J}_{A_{00}} : I \to \mathcal{U}(A,A)(1_A,1_A) = j_{1_A}.$$

3 The 3–category of enriched 2–categories

As in [Forcey, 2003], I now describe the (strict) 3–category $\mathcal{V}$–2–Cat (or $\mathcal{V}$–Cat–Cat) whose objects are (strict, small) $\mathcal{V}$–2–categories.

Definition 11 *For two $\mathcal{V}$–2–categories $\mathcal{U}$ and $\mathcal{W}$ a $\mathcal{V}$–2–functor $T : \mathcal{U} \to \mathcal{W}$ is a function on objects $|\mathcal{U}| \to |\mathcal{W}|$ and a family of $\mathcal{V}$–functors $T_{UU'} : \mathcal{U}(U, U') \to \mathcal{W}(TU, TU')$. These latter obey commutativity of the usual diagrams.*

1. For $U, U', U'' \in |\mathcal{U}|$

$$
\begin{array}{ccc}
\bullet & \xrightarrow{\ \mathcal{M}_{UU'U''}\ } & \bullet \\
{\scriptstyle T_{U'U''}\otimes_1^{(1)}T_{UU'}}\Big\downarrow & & \Big\downarrow{\scriptstyle T_{UU''}} \\
\bullet & \xrightarrow[\ \mathcal{M}_{(TU)(TU')(TU'')}\ ]{} & \bullet
\end{array}
$$

2.

$$
\begin{array}{ccc}
& & \bullet \\
& {\scriptstyle \mathcal{J}_U}\nearrow & \Big\downarrow{\scriptstyle T_{UU}} \\
\mathcal{I} & \searrow_{\ \mathcal{J}_{TU}} & \\
& & \bullet
\end{array}
$$

For objects this means that $T_{U'U''}(f)T_{UU'}(g) = T_{UU''}(fg)$ and $T_{UU}(1_U) = 1_{TU}$. The reader should unpack both diagrams into terms of hom–object morphisms and $\mathcal{V}$–functoriality. The fact that the hom–object morphisms are actually hom–category $\mathcal{V}$–functors corresponds to the need for $\mathcal{V}$–2–functors to preserve all the structure that exists, including the vertical composition.

Definition 12 *A $\mathcal{V}$–2–natural transformation $\alpha : T \to S : \mathcal{U} \to \mathcal{W}$ is a function sending each $U \in |\mathcal{U}|$ to a $\mathcal{V}$–functor $\alpha_U : \mathcal{I} \to \mathcal{W}(TU, SU)$ in such a way that we have commutativity of*

$$
\begin{array}{ccccc}
& \mathcal{I} \otimes_1^{(1)} \mathcal{U}(U, U') & \xrightarrow{\ \alpha_{U'}\otimes_1^{(1)}T_{UU'}\ } & \mathcal{W}(TU', SU') \otimes_1^{(1)} \mathcal{W}(TU, TU') & \\
{\scriptstyle =}\nearrow & & & & \searrow^{\ \mathcal{M}} \\
\mathcal{U}(U, U') & & & & \mathcal{W}(TU, SU') \\
{\scriptstyle =}\searrow & & & & \nearrow_{\ \mathcal{M}} \\
& \mathcal{U}(U, U') \otimes_1^{(1)} \mathcal{I} & \xrightarrow[\ S_{UU'}\otimes_1^{(1)}\alpha_U\ ]{} & \mathcal{W}(SU, SU') \otimes_1^{(1)} \mathcal{W}(TU, SU) &
\end{array}
$$

Unpacking this a bit, we see that α_U is an object $q = \alpha_U(0)$ in the $\mathcal{V}$–category $\mathcal{W}(TU, SU)$ and a morphism $\alpha_{U_{00}} : I \to \mathcal{W}(TU, SU)(q, q)$. By the $\mathcal{V}$–functoriality of α_U we see that $\alpha_{U_{00}} = j_q$. The axiom then states that $q'T_{UU'}(f) = S_{UU'}(f)q$ for all f, and that

$$\mathcal{M}_{(TU)(TU')(SU')_{(q',T_{UU'}(f))(q',T_{UU'}(g))}} \circ (j_{q'} \otimes_2 T_{UU'_{fg}}) = \mathcal{M}_{(TU)(SU)(SU')_{(S_{UU'}(f),q)(S_{UU'}(g),q)}} \circ (S_{UU'_{fg}} \otimes_2 j_q)$$

The heuristic picture (following the pattern set in the definition of a $\mathcal{V}$–2–category) is as follows:

$$
\begin{array}{ccccc}
& \xrightarrow{\;Tf\;} & & & \\
TU & & TU' \xrightarrow{\;\;q'\;\;} SU' & = & TU \xrightarrow{\;\;q\;\;} SU \\
& \xrightarrow{\;Tg\;} & & &
\end{array}
\qquad
\begin{array}{c}
\xrightarrow{Sf} \\
SU' \\
\xrightarrow{Sg}
\end{array}
$$

Definition 13 *Given two $\mathcal{V}$–2–natural transformations a $\mathcal{V}$–modification between them $\mu : \theta \to \phi : T \to S : \mathcal{U} \to \mathcal{W}$ is a function that sends each object $U \in |\mathcal{U}|$ to a morphism $\mu_U : I \to \mathcal{W}(TU, SU)(\theta_U(0), \phi_U(0))$ in such a way that the following diagram commutes. (Let $\theta_U(0) = q$, $\phi_U(0) = \hat{q}$, $\theta_{U'}(0) = q'$ and $\phi_{U'}(0) = \hat{q}'$.)*

$$
\mathcal{W}(TU', SU')(q', \hat{q}') \otimes_2 \mathcal{W}(TU, TU')(T_{UU'}(f), T_{UU'}(g))
$$

$$
\xrightarrow{\quad\mu_{U'} \otimes_2 T_{UU'_{fg}}\quad} \qquad\qquad \xrightarrow{\quad\mathcal{M}\quad}
$$

$$
I \otimes_2 \mathcal{U}(U, U')(f, g) \qquad\qquad \mathcal{W}(TU, SU')(q'T_{UU'}(f), \hat{q}'T_{UU'}(g))
$$

$$
= \bigg\uparrow \qquad\qquad\qquad\qquad \bigg\|
$$

$$
\mathcal{U}(U, U')(f, g)
$$

$$
= \bigg\downarrow
$$

$$
\mathcal{U}(U, U')(f, g) \otimes_2 I \qquad\qquad \mathcal{W}(TU, SU')(S_{UU'}(f)q, S_{UU'}(g)\hat{q})
$$

$$
\xrightarrow{\quad S_{UU'_{fg}} \otimes_2 \mu_U\quad} \qquad\qquad \xrightarrow{\quad\mathcal{M}\quad}
$$

$$
\mathcal{W}(SU, SU')(S_{UU'}(f), S_{UU'}(g)) \otimes_2 \mathcal{W}(TU, SU)(q, \hat{q})
$$

Notice that since $\theta_{U_{00}} = j_{\theta_U(0)}$ for all $\mathcal{V}$–2–natural transformations θ we have that the morphism μ_U seen as a "family" consisting of a single morphism (corresponding to $0 \in |\mathcal{I}|$) constitutes a $\mathcal{V}$–natural transformation from θ_U to ϕ_U. Occasionally I reflect this by denoting μ_U as μ_{U0}.

The heuristic picture here is:

$$TU \xrightarrow[Tg]{Tf} TU' \xrightarrow[\hat{q}']{q'} SU' \quad = \quad TU \xrightarrow{q} SU \xrightarrow[Sg]{Sf} SU'$$

Theorem 1 $\mathcal{V}$–2–categories, $\mathcal{V}$–2–functors, $\mathcal{V}$–2–natural transformations and $\mathcal{V}$–modifications form a 3–category called $\mathcal{V}$–2–Cat.

Proof (Part 1.) Recall that a 3–category is a category enriched over 2–Cat. This is expanded in terms of axioms in [Borceux, 1994]. Our objects are $\mathcal{V}$–2–categories. There are two parts of the proof. In part 1 we show that for every pair $\mathcal{U}, \mathcal{W}$ of $\mathcal{V}$–2–categories we have a 2–category made up of $\mathcal{V}$–2–functors, $\mathcal{V}$–2–natural transformations and $\mathcal{V}$–modifications. For now then $\mathcal{V}$–2–functors are the 0–cells, $\mathcal{V}$–2–natural transformations are the 1–cells, and $\mathcal{V}$–modifications are the 2–cells as in the following picture.

Throughout I will use the following notation: Composition along a $\mathcal{V}$–2–natural transformation will be indicated with "$\circ$." Composition along a $\mathcal{V}$–2–functor will be indicated with "$*$." Composition along a $\mathcal{V}$–2–category will be indicated by juxtaposition.

Composition of $\mathcal{V}$-2–natural transformations $\gamma : T \to S$ and $\beta : S \to R$ along a $\mathcal{V}$–2–functor S is given by

$$(\beta * \gamma)_U = \quad \mathcal{I} = \mathcal{I} \otimes_1^{(1)} \mathcal{I}$$
$$\downarrow \beta_U \otimes_1^{(1)} \gamma_U$$
$$\mathcal{W}(SU, RU) \otimes_1^{(1)} \mathcal{W}(TU, SU)$$
$$\downarrow \mathcal{M}$$
$$\mathcal{W}(TU, RU)$$

Let $\beta_U(0) = \hat{q}$ and $\gamma_U(0) = \check{q}$. By expanding the definition of this composition we see that $(\beta * \gamma)_U(0) = \hat{q}\check{q}$ and that $(\beta * \gamma)_{U_{00}} = j_{\hat{q}\check{q}}$

Since it prefigures a similar proof for $\mathcal{V}$–modifications, I include the proof that this composition forms a valid $\mathcal{V}$–2–natural transformation even though it follows closely the analogous proof for $\mathcal{V}$–natural transformations as in [Eilenberg and Kelly, 1965]. For $\beta * \gamma$ to be a $\mathcal{V}$–2–natural transformation the exterior of the following diagram must commute.

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$$\mathcal{W}(TU, RU')$$

$$\mathcal{W}(TU', RU') \otimes_1^{(1)} \mathcal{W}(TU, TU')$$

$$\mathcal{W}(SU', RU') \otimes_1^{(1)} \mathcal{W}(TU, SU')$$

$$\mathcal{W}(SU, RU') \otimes_1^{(1)} \mathcal{W}(TU, SU)$$

$$\mathcal{W}(RU, RU') \otimes_1^{(1)} \mathcal{W}(TU, RU)$$

$$(\mathcal{W}(SU', RU') \otimes_1^{(1)} \mathcal{W}(TU', SU')) \otimes_1^{(1)} \mathcal{W}(TU, TU')$$

$$\mathcal{W}(RU, RU') \otimes_1^{(1)} (\mathcal{W}(SU, RU) \otimes_1^{(1)} \mathcal{W}(TU, SU))$$

$$(\mathcal{I} \otimes_1^{(1)} \mathcal{I}) \otimes_1^{(1)} \mathcal{U}(U, U')$$

$$\mathcal{U}(U, U') \otimes_1^{(1)} (\mathcal{I} \otimes_1^{(1)} \mathcal{I})$$

$$\mathcal{I} \otimes_1^{(1)} \mathcal{U}(U, U')$$

$$\mathcal{U}(U, U') \otimes_1^{(1)} \mathcal{I}$$

$$\mathcal{U}(U, U')$$

Arrow labels: $\mathcal{M}$, $\mathcal{M} \otimes_1^{(1)} 1$, $1 \otimes_1^{(1)} \mathcal{M}$, $\alpha^{(1)1}$, $(\beta_{U'} \otimes_1^{(1)} \gamma_{U'}) \otimes_1^{(1)} \tau_{TUU'}$, $\beta_{U'} \otimes_1^{(1)} 1$, $1 \otimes_1^{(1)} (\gamma_{U'} \otimes_1^{(1)} \tau_{UU'})$, $1 \otimes_1^{(1)} (S_{UU'} \otimes_1^{(1)} \gamma_U)$, $(\beta_{U'} \otimes_1^{(1)} S_{UU'}) \otimes_1^{(1)} 1$, $(R_{UU'} \otimes_1^{(1)} \beta_U) \otimes_1^{(1)} 1$, $1 \otimes_1^{(1)} \gamma_U$, $R_{UU'} \otimes_1^{(1)} (\beta_U \otimes_1^{(1)} \gamma_U)$, $=$

The arrows marked with an "=" all occur as copies of $\mathcal{I}$ are tensored to the object at the arrow's source. The 3 leftmost regions commute by the naturality of $\alpha^{(1)1} = \alpha^2$. The 2 embedded central hexagons commute by the definition of $\mathcal{V}$–2–natural transformations for γ and β. The three pentagons on the right are copies of the pentagon axiom for the composition $\mathcal{M}$. The associativity of this composition also follows directly from the latter axiom.

The identities for this composition are $\mathcal{V}$–2–natural transformations $\mathbf{1}_T :$ $T \rightarrow T$ where $(\mathbf{1}_T)_U = \mathcal{J}_{TU}$. That this describes a 2-sided identity for the composition above is easily checked using the unit axioms for a $\mathcal{V}$–2–category.

The composition of two $\mathcal{V}$–modifications along a $\mathcal{V}$–2–transformation is given by the composition of the underlying $\mathcal{V}$–natural transformations. So given $\mathcal{V}$–2–natural transformations α, β and $\sigma : F \rightarrow G : \mathcal{U} \rightarrow \mathcal{W}$, and $\mathcal{V}$–modifications $\mu : \alpha \rightarrow \beta$ and $\nu : \beta \rightarrow \sigma$ as in the following picture

where $\alpha_U(0) = q$, $\beta_U(0) = \hat{q}$ and $\sigma_U(0) = \check{q}$, we have

$$(\nu \circ \mu)_U \;=\; \begin{array}{c} I = I \otimes_1 I \\ \downarrow{\scriptstyle \nu_U \otimes_1 \mu_U} \\ \mathcal{W}(FU, GU)(\hat{q}, \check{q}) \otimes_1 \mathcal{W}(FU, GU)(q, \hat{q}) \\ \downarrow{\scriptstyle M} \\ \mathcal{W}(FU, GU)(q, \check{q}) \end{array}$$

We see that this composition is associative by the associativity pentagon for M. We also see that the result of a composition is a $\mathcal{V}$–natural transformation as well. It needs to be checked that the result of a composition is a valid $\mathcal{V}$–modification. This is seen by showing that the exterior of the following diagram commutes.

The first bullet in the following diagram is $\mathcal{U}(U, U')(f, g)$. Other objects include:

$$A = (I \otimes_1 I) \otimes_2 \mathcal{U}(U, U')(f, g) \;\; ; \; B = (I \otimes_1 I) \otimes_2 (\mathcal{U}(U, U')(f, g) \otimes_1 I) \; ;$$

$$C = (I \otimes_1 I) \otimes_2 (\mathcal{U}(U, U')(f, g) \otimes_1 \mathcal{U}(U, U')(f, f)) \;\; ; \; D = (I \otimes_2 \mathcal{U}(U, U')(f, g)) \otimes_1 (I \otimes_2 \mathcal{U}(U, U')(f, f)) \; ;$$

$$E = \mathcal{U}(U, U')(f, g) \otimes_1 I \;\; ; \; H = \mathcal{U}(U, U')(f, g) \otimes_1 \mathcal{U}(U, U')(f, f) \; ;$$

$$K = \mathcal{U}(U, U')(f, g) \otimes_2 (I \otimes_1 I) \;\; ; \; L = (\mathcal{U}(U, U')(f, g) \otimes_1 I) \otimes_2 (I \otimes_1 I) \; ;$$

$$N = (\mathcal{U}(U, U')(f, g) \otimes_2 I) \otimes_1 (\mathcal{U}(U, U')(f, f) \otimes_2 I) \;\; ;$$

$$P = (\mathcal{U}(U, U')(f, g) \otimes_2 \mathcal{U}(U, U')(f, f)) \otimes_1 (\mathcal{U}(U, U')(f, f) \otimes_2 I).$$

Commutative diagram with nodes A, B, C, D, E, H, K, L, N, P, M and unlabeled bullet vertices.

Edge and region labels:

$(\nu_{U'} \otimes_1 \mu_{U'}) \otimes_2 F_{UU'}$

$M \otimes_2 1$

$M \otimes_2 M$

$\mathcal{M}$

$\mathcal{M}$

$(1 \otimes 1) \otimes_2 (1 \otimes_1 j F_{UU'}(f))$

$\eta^{1,2}$

$\mathcal{M} \otimes_1 \mathcal{M}$

$(\nu_{U'} \otimes_1 \mu_{U'}) \otimes_2 (F_{UU'} \otimes_1 F_{UU'})$

$(\nu_{U'} \otimes_2 F_{UU'}) \otimes_1 (\mu_{U'} \otimes_2 F_{UU'})$

$(\nu_{U'} \otimes_1 \mu_{U'}) \otimes_2 (F_{UU'} \otimes_1 1)$

$(1 \otimes_1 1) \otimes_2 (1 \otimes_1 j_f)$

$\eta^{1,2}$

$1 \otimes_1 j_f$

$(G_{UU'} \otimes_2 \nu_U) \otimes_1 (G_{UU'} \otimes_2 \mu_U)$

$\mathcal{M} \otimes_1 \mathcal{M}$

$(G_{UU'} \otimes_1 G_{UU'}) \otimes_2 (\nu_U \otimes_1 \mu_U)$

$(1 \otimes_1 j_f) \otimes_2 (1 \otimes_1 1)$

$\eta^{1,2}$

$(1 \otimes_1 j G_{UU'}(f)) \otimes_2 (1 \otimes_1 1)$

$M \otimes_2 M$

$1 \otimes_2 M$

$(G_{UU'} \otimes_1 1) \otimes_2 (\nu_U \otimes_1 \mu_U)$

$G_{UU'} \otimes_2 (\nu_U \otimes_1 \mu_U)$

$\mathcal{M}$

$\mathcal{M}$

Region labels: [a], [b], [c], [d], [e], [f], [g], [h], [i], [j], [k], [l], [m]

Equality arrows: $=$

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The arrows marked with an "=" all occur as copies of I are tensored to the object at the arrow's source. Therefore the quadrilateral regions [a],[e],[f],[h],[i] and [j] all commute trivially. The uppermost and lowermost quadrilaterals commute by the property of composing with units in an enriched category. The two triangles commute by the external unit condition for iterated monoidal categories. Regions [b] and [k] commute by respect of units by enriched functors. Regions [c] and [l] commute by naturality of η. Region [g] commutes by the definition of $\mathcal{V}$–modification for ν and μ. Regions [d] and [m] commute by the $\mathcal{V}$–functoriality of $\mathcal{M}$. The following heuristic diagram for this proof is quite instructive. (See the pattern set for these diagrams in the definition of a $\mathcal{V}$–2–category.)

Thus identities $\mathbf{1}_\alpha$ for this composition are families of $\mathcal{V}$–natural equivalences. Since α_U is a $\mathcal{V}$–functor from $\mathcal{I}$ to $\mathcal{W}(TU, SU)$ this means specifically that $((\mathbf{1}_\alpha)_U)_0 = j_{\alpha_U(0)} = j_q$. Recall that here the "family" has only one member, corresponding to the single object in $\mathcal{I}$. That this describes a 2-sided identity for the composition above is easily checked using the unit axioms for a $\mathcal{V}$–category.

In order to define composition of all allowable pasting diagrams in the 2-category, we need only to define the composition described by left and right whiskering diagrams (as partial functors) and check that these can be combined into a well–defined horizontal composition. The first picture shows a 1-cell (that is a $\mathcal{V}$–2–natural transformation between $\mathcal{V}$–2–functors $F, G : \mathcal{U} \to \mathcal{W}$) following a 2-cell (a $\mathcal{V}$–modification). These are composed to form a new 2-cell as follows

where $\gamma * \psi$ and $\gamma * \beta$ are described above, and $\gamma * \mu$ has components given by the following composition: (Let $\psi_U(0) = q$, $\beta_U(0) = \hat{q}$ and $\gamma_U(0) = \check{q}$. Note that

$j_{\check{q}} = \gamma_{U_{00}}.)$

$$(\gamma * \mu)_U \quad = \qquad\qquad I = I \otimes_2 I$$

$$\downarrow j_{\check{q}} \otimes_2 \mu_U$$

$$\boldsymbol{\mathcal{W}}(GU, HU)(\check{q}, \check{q}) \otimes_2 \boldsymbol{\mathcal{W}}(FU, GU)(q, \hat{q})$$

$$\downarrow \mathcal{M}$$

$$\boldsymbol{\mathcal{W}}(FU, HU)(\check{q}q, \check{q}\hat{q})$$

For this composition to yield a valid $\mathcal{V}$–modification the exterior of the following diagram must commute.

Commutative diagram (rotated 90°):

Top row nodes (left to right):
- $(\mathcal{W}(GU',HU')(\hat{q}',\hat{q}')\otimes_2\mathcal{W}(FU',GU')(\check{q}',\hat{q}'))\otimes_2\mathcal{W}(FU,FU')(Ff,Fg)$
- $\mathcal{W}(FU',HU')(\check{q}'\hat{q}',\check{q}'\hat{q}')\otimes_2\mathcal{W}(FU,FU')(Ff,Fg)$
- $\mathcal{W}(FU',HU')(\check{q}'\hat{q}'Ff,\check{q}'\hat{q}'Fg)$
- $\mathcal{W}(FU,HU')(\check{q}'Gfq,\check{q}'Gg\hat{q})$
- $\mathcal{W}(FU,HU')(Hf\hat{q},Hg\hat{q})$

Middle nodes:
- $\mathcal{W}(GU',HU')(\check{q}',\hat{q}')\otimes_2\mathcal{W}(FU,GU')(q'Ff,\hat{q}'Fg)$
- $\mathcal{W}(GU',HU')(\hat{q}',\hat{q}')\otimes_2\mathcal{W}(FU,GU')(Gfq,Gg\hat{q})$
- $\mathcal{W}(GU,HU')(\check{q}'Gf,\hat{q}'Gg)\otimes_2\mathcal{W}(FU,GU')(q,\hat{q})$
- $\mathcal{W}(GU,HU')(Hf\hat{q},Hg\hat{q})\otimes_2\mathcal{W}(FU,GU')(q,\hat{q})$
- $\mathcal{W}(HU,HU')(Hf,Hg)\otimes_2\mathcal{W}(FU,HU')(\hat{q},\hat{q})$

Bottom nodes:
- $(I\otimes_2 I)\otimes_2\mathcal{U}(U,U')(f,g)$
- $I\otimes_2\mathcal{U}(U,U')(f,g)$
- $\mathcal{U}(U,U')(f,g)$
- $\mathcal{U}(U,U')(f,g)\otimes_2 I$
- $\mathcal{U}(U,U')(f,g)\otimes_2(I\otimes_2 I)$
- $\mathcal{W}(HU,HU')(Hf,Hg)\otimes_2(\mathcal{W}(GU,HU')(\check{q},\hat{q})\otimes_2\mathcal{W}(FU,GU)(q,\hat{q}))$

Arrow labels:
- α^2
- $\mathcal{M}\otimes 1$
- $1\otimes_2\mathcal{M}$
- $\mathcal{M}$
- $(j_{q'}\otimes_2\mu_{U'})\otimes_2 F_{UU'}$
- $1\otimes_2(\mu_{U'}\otimes_2 F_{UU'})$
- $1\otimes_2(G_{UU'}\otimes_2\mu_U)$
- $(j_{q'}\otimes_2 G_{UU'})\otimes 1$
- $(H_{UU'}\otimes_2 j_{\hat{q}})\otimes 1$
- $H_{UU'}\otimes_2(j_{\hat{q}}\otimes_2\mu_U)$
- $j_{j_{q'}}\otimes_2 1$
- $1\otimes_2\mu_U$
- $=$

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The arrows marked with an "=" all occur as copies of I are tensored to the object at the arrow's source. The 3 leftmost regions commute by the naturality of α^2. The 2 embedded central "hexagons" commute by the definition of $\mathcal{V}$–modifications for μ and $\mathbf{1}_\gamma$. The three pentagons on the right are copies of the pentagon axiom for the composition $\mathcal{M}$.

The second picture shows a 2–cell following a 1–cell. These are composed as follows

$$E \xrightarrow{\rho} F \underset{\beta}{\overset{\psi}{\Rightarrow}} G \qquad \text{is composed to become} \qquad E \underset{\beta*\rho}{\overset{\psi*\rho}{\Rightarrow}} G$$

where $\mu*\rho$ has components given by the following composition: (Let $\rho_U(0) = \overline{q}$. Note that $j_{\overline{q}} = \rho_{U_{00}}$.)

$$
(\mu * \rho)_U \;\; = \;\;
\begin{array}{c}
I = I \otimes_2 I \\
\downarrow{\scriptstyle \mu_U \otimes_2 j_{\overline{q}}} \\
\mathcal{W}(FU, GU)(q, \hat{q}) \otimes_2 \mathcal{W}(EU, FU)(\overline{q}, \overline{q}) \\
\downarrow{\scriptstyle \mathcal{M}} \\
\mathcal{W}(FU, HU)(q\overline{q}, \hat{q}\overline{q})
\end{array}
$$

For this composition to yield a valid $\mathcal{V}$–modification the exterior of the following diagram must commute.
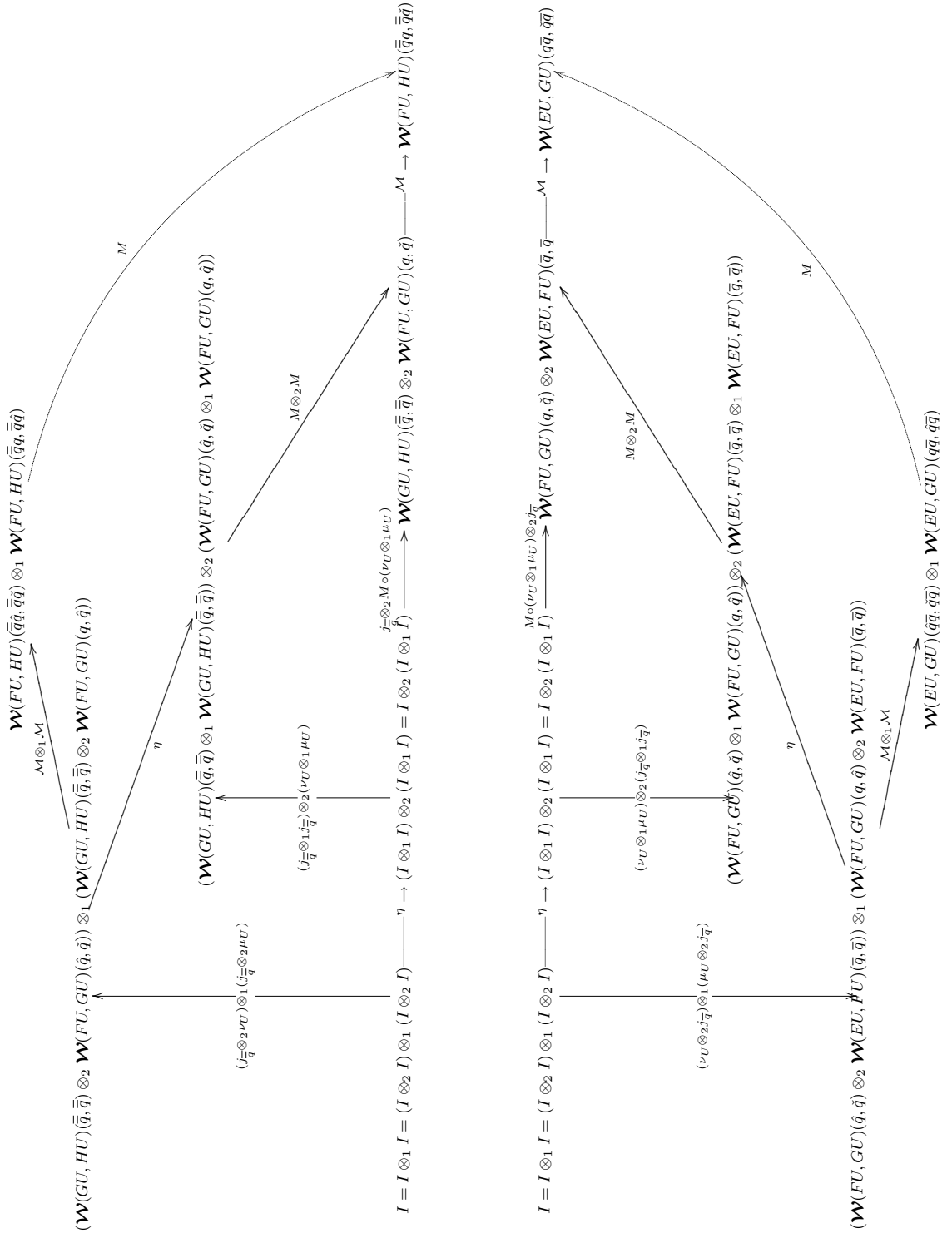
The arrows marked with an "=" all occur as copies of I are tensored to the object at the arrow's source. The 3 leftmost regions commute by the naturality of α^2. The 2 embedded central "hexagons" commute by the definition of $\mathcal{V}$–modifications for μ and $\mathbf{1}_\rho$. The three pentagons on the right are copies of the pentagon axiom for the composition $\mathcal{M}$.

What we have developed here are the partial functors of the composition morphism implicit in enriching over **Cat**. The said composition morphism is a functor of two variables. That the partial functors can be combined to make the functor of two variables is implied by the commutativity of a diagram that describes the two ways of combining them (see [Mac Lane, 1998]). One thing that needs to be checked is that composing horizontally adjacent 2–cells is well–defined. We also need to check that the partial functors are indeed functorial. This is shown by checking that the whiskering distributes over the vertical composition, and checking that whiskering is the same as horizontally composing with an identity 2–cell. (The latter is actually showing more than that whiskering onto an identity 2–cell is the same as horizontally composing two identity 2–cells, which in turn is more than what we really need: i.e. whiskering onto an identity 2–cell gives an identity 2–cell for the composed 1–cells. It is often however, just as convenient to prove.) I start with the first axiom of functoriality.

First we need to check that the whiskering distributes, i.e. that $(\rho * \nu) \circ (\rho * \mu) = \rho * (\nu \circ \mu)$ and that $(\nu * \xi) \circ (\mu * \xi) = (\nu \circ \mu) * \xi$ as in the following picture.
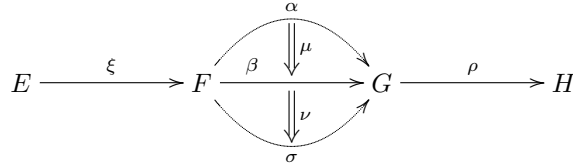
This requires the exteriors of the following two diagrams to commute (Let $\xi_U(0) = \overline{q}$, $\rho_U(0) = \overline{\overline{q}}$, $\alpha_U(0) = q$, $\beta_U(0) = \hat{q}$ and $\sigma_U(0) = \check{q}$,)

Top diagram

$$\mathcal{W}(FU, HU)(\bar{\bar{q}}\hat{q}, \bar{\bar{q}}\hat{q}) \otimes_1 \mathcal{W}(FU, HU)(\bar{\bar{q}}q, \bar{\bar{q}}\hat{q})$$

$$(\mathcal{W}(GU, HU)(\bar{\bar{q}}, \bar{q}) \otimes_2 \mathcal{W}(FU, GU)(\hat{q}, \hat{q})) \otimes_1 (\mathcal{W}(GU, HU)(\bar{\bar{q}}, \bar{q}) \otimes_2 \mathcal{W}(FU, GU)(q, \hat{q}))$$

$$(\mathcal{W}(GU, HU)(\bar{\bar{q}}, \bar{q}) \otimes_1 \mathcal{W}(GU, HU)(\bar{\bar{q}}, \bar{q})) \otimes_2 (\mathcal{W}(FU, GU)(\hat{q}, \hat{q}) \otimes_1 \mathcal{W}(FU, GU)(q, \hat{q}))$$

$$\mathcal{W}(GU, HU)(\bar{\bar{q}}, \bar{q}) \otimes_2 \mathcal{W}(FU, GU)(\hat{q}, \hat{q}) \xrightarrow{\;\mathcal{M}\;} \mathcal{W}(FU, HU)(\bar{\bar{q}}\hat{q}, \bar{\bar{q}}\hat{q})$$

$$I = I \otimes_1 I = (I \otimes_2 I) \otimes_1 (I \otimes_2 I)$$

$$I = I \otimes_1 I = I \otimes_2 (I \otimes_1 I) = I \otimes_2 (I \otimes_1 I) \xrightarrow{\;\eta\;}$$

Arrow labels: $\mathcal{M} \otimes_1 \mathcal{M}$, M, η, $M \otimes_2 M$, $j_{\bar{\bar{q}}} \otimes_2 M \circ (\nu_U \otimes_1 \mu_U)$, $(j_{\bar{\bar{q}}} \otimes_2 \nu_U) \otimes_1 (j_{\bar{\bar{q}}} \otimes_2 \mu_U)$, $(j_{\bar{\bar{q}}} \otimes_1 j_{\bar{\bar{q}}}) \otimes_2 (\nu_U \otimes_1 \mu_U)$

Bottom diagram

$$\mathcal{W}(EU, GU)(q\hat{q}, q\hat{q}) \otimes_1 \mathcal{W}(EU, GU)(q\hat{q}, q\hat{q})$$

$$(\mathcal{W}(FU, GU)(\hat{q}, \hat{q}) \otimes_2 \mathcal{W}(EU, FU)(\bar{q}, \bar{q})) \otimes_1 (\mathcal{W}(FU, GU)(\hat{q}, \hat{q}) \otimes_2 \mathcal{W}(EU, FU)(\bar{q}, \bar{q}))$$

$$\mathcal{W}(FU, GU)(q, \hat{q}) \otimes_2 \mathcal{W}(EU, FU)(\bar{q}, \bar{q}) \xrightarrow{\;\mathcal{M}\;} \mathcal{W}(EU, GU)(q\hat{q}, q\hat{q})$$

$$\mathcal{W}(EU, GU)(q\hat{q}, q\hat{q}) \otimes_1 \mathcal{W}(EU, GU)(q\hat{q}, q\hat{q})$$

$$I = I \otimes_1 I = (I \otimes_2 I) \otimes_1 (I \otimes_2 I)$$

$$I = I \otimes_1 I = I \otimes_2 (I \otimes_1 I) = I \otimes_2 (I \otimes_1 I) \xrightarrow{M \circ (\nu_U \otimes_1 \mu_U) \otimes_2 j_{\bar{q}}}$$

Arrow labels: $\mathcal{M} \otimes_1 \mathcal{M}$, M, η, $M \otimes_2 M$, $(\nu_U \otimes_1 \mu_U) \otimes_2 (j_{\bar{q}} \otimes_1 j_{\bar{q}})$, $(\nu_U \otimes_2 j_{\bar{q}}) \otimes_1 (\mu_U \otimes_2 j_{\bar{q}})$, $M \circ (\nu_U \otimes_1 \mu_U) \otimes_2 j_{\bar{q}}$

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These commute since the interior regions all commute. The leftmost quadrilaterals commute by naturality of η. The central triangular regions commute by the unit axioms of $\mathcal{V}$–categories. The pentagonal regions commute by the $\mathcal{V}$–functoriality of $\mathcal{M}$.

This commutativity has verified that the partial functors of the horizontal composition functor (whiskers) in fact do respect the composition in their domain.
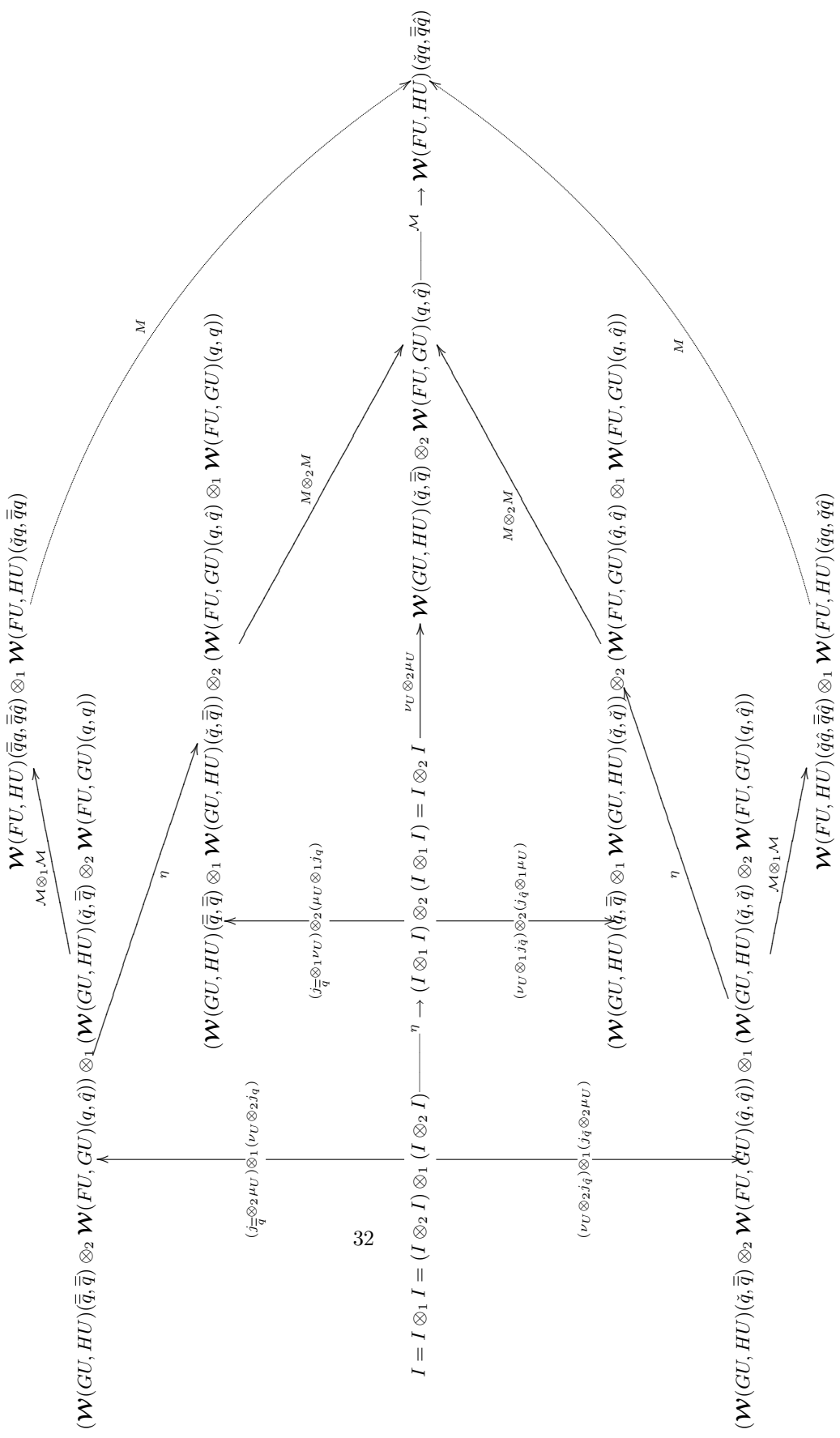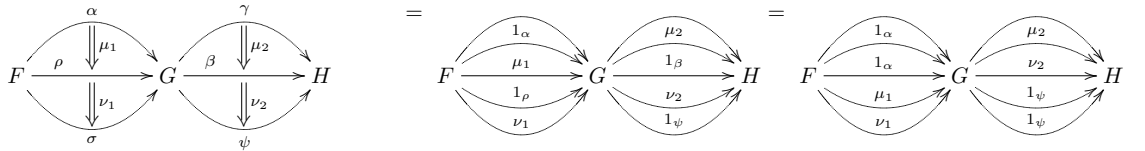
We still need the two ways of composing the below cells using whiskers to be equivalent:

That is, we need:

$$\nu * \mu = (\sigma * \mu) \circ (\nu * \psi) = (\nu * \beta) \circ (\gamma * \mu).$$

In terms of the above definitions, the exterior of the following diagram must commute (Let $\rho_U(0) = \bar{\bar{q}}$.)

$$\mathbf{W}(FU, HU)(\check{q}q, \overline{\overline{q}}\hat{q})$$

$$I = I \otimes_1 I = (I \otimes_2 I) \otimes_1 (I \otimes_2 I) \xrightarrow{\ \eta\ } (I \otimes_1 I) \otimes_2 (I \otimes_1 I) = I \otimes_2 I \xrightarrow{\ \nu_U \otimes_2 \mu_U\ } \mathbf{W}(GU, HU)(\check{q}, \overline{q}) \otimes_2 \mathbf{W}(FU, GU)(q, \hat{q}) \xrightarrow{\ \mathcal{M}\ } \mathbf{W}(FU, HU)(\check{q}q, \overline{\overline{q}}\hat{q})$$

Arrow labels:

M (left arc), $\quad M$ (right arc)

$\mathcal{M} \otimes_1 \mathcal{M}$

η

$M \otimes_2 M$

$M \otimes_2 M$

η

$\mathcal{M} \otimes_1 \mathcal{M}$

$(j\tfrac{=}{q} \otimes_1 \nu_U) \otimes_2 (\mu_U \otimes_1 j_q)$

$(j\tfrac{=}{q} \otimes_2 \mu_U) \otimes_1 (\nu_U \otimes_2 j_q)$

$(\nu_U \otimes_1 j_{\hat{q}}) \otimes_2 (j_{\hat{q}} \otimes_1 \mu_U)$

$(\nu_U \otimes_2 j_{\hat{q}}) \otimes_1 (j_{\hat{q}} \otimes_2 \mu_U)$

Nodes:

$(\mathbf{W}(FU, HU)(\overline{\overline{q}}q, \overline{\overline{q}}\hat{q}) \otimes_1 \mathbf{W}(FU, HU)(\check{q}q, \overline{\overline{q}}q)$

$(\mathbf{W}(GU, HU)(\overline{\overline{q}}, \overline{q}) \otimes_2 \mathbf{W}(FU, GU))(\check{q}, \overline{q}) \otimes_2 \mathbf{W}(FU, GU)(q, q))$

$(\mathbf{W}(GU, HU)(\overline{\overline{q}}, \overline{q}) \otimes_1 \mathbf{W}(GU, HU)(\check{q}, \overline{\overline{q}})) \otimes_2 \mathbf{W}(FU, GU)(q, \hat{q}) \otimes_1 \mathbf{W}(FU, GU)(q, q))$

$\mathbf{W}(GU, HU)(\overline{\overline{q}}, \overline{q}) \otimes_1 \mathbf{W}(GU, HU)(\check{q}, \overline{\overline{q}}) \otimes_2 \mathbf{W}(FU, GU)(q, \hat{q}) \otimes_1 \mathbf{W}(FU, GU)(q, q))$

$(\mathbf{W}(GU, HU)(\check{q}, \overline{q}) \otimes_1 \mathbf{W}(GU, HU)(\hat{q}, \hat{q})) \otimes_2 (\mathbf{W}(FU, GU)(\hat{q}, \hat{q}) \otimes_1 \mathbf{W}(FU, GU)(q, \hat{q}))$

$(\mathbf{W}(GU, HU)(\check{q}, \overline{q}) \otimes_2 \mathbf{W}(FU, GU)(\hat{q}, \hat{q})) \otimes_1 (\mathbf{W}(GU, HU)(\hat{q}, \hat{q}) \otimes_2 \mathbf{W}(FU, GU)(q, \hat{q}))$

$\mathbf{W}(FU, HU)(\hat{q}\hat{q}, \overline{\overline{q}}\hat{q}) \otimes_1 \mathbf{W}(FU, HU)(\check{q}q, \hat{q}\hat{q})$

This commutes since the interior regions all commute. The leftmost quadrilaterals commute by naturality of η. The central triangular regions commute by the unit axioms of $\mathcal{V}$–categories. The upper and lower pentagonal regions commute by the $\mathcal{V}$–functoriality of $\mathcal{M}$. This composition gives a valid $\mathcal{V}$–modification since the whiskered pieces are valid and since the composition along a $\mathcal{V}$–2–natural transformation gives a valid $\mathcal{V}$–modification. The central leg of the above diagram gives a more direct description of the composition of $\mathcal{V}$–2–modifications along a $\mathcal{V}$–2–functor. From this description it is automatic that whiskering a $\mathcal{V}$–2–natural transformation to a $\mathcal{V}$–modification along a $\mathcal{V}$–2–functor is the same as composing along that $\mathcal{V}$–2–functor with an identity $\mathcal{V}$–modification corresponding to the whisker. We also see from this description that the associativity of this composition follows immediately from the associativity axiom of a strict $\mathcal{V}$–2–category.

Now we can show the functoriality of the entire composition functor. (The general proof regarding partial functors of a functor of two variables is in [Mac Lane, 1998].) This states that, in the following picture, $(\nu_2 * \nu_1) \circ (\mu_2 * \mu_1) = (\nu_2 \circ \mu_2) * (\nu_1 \circ \mu_1)$.
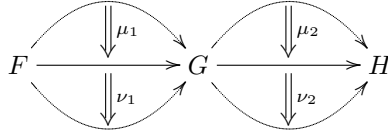
This is shown by using what we have established. The following series of pictures serve to illustrate the proof.

Or symbolically:
$$(\nu_2 * \nu_1) \circ (\mu_2 * \mu_1)$$
$$= ((\psi * \nu_1) \circ (\nu_2 * \rho)) \circ ((\beta * \mu_1) \circ (\mu_2 * \alpha))$$
$$= (\psi * \nu_1) \circ ((\nu_2 * \rho) \circ (\beta * \mu_1)) \circ (\mu_2 * \alpha)$$
$$= (\psi * \nu_1) \circ ((\psi * \mu_1) \circ (\nu_2 * \alpha)) \circ (\mu_2 * \alpha)$$
$$= ((\psi * \nu_1) \circ (\psi * \mu_1)) \circ ((\nu_2 * \alpha) \circ (\mu_2 * \alpha))$$
$$= (\psi * (\nu_1 \circ \mu_1)) \circ ((\nu_2 \circ \mu_2) * \alpha)$$
$$= (\nu_2 \circ \mu_2) * (\nu_1 \circ \mu_1)$$

The exchange identity is precisely the functoriality (respect of vertical composition) of the functor of two variables that describes the horizontal composition. We also have the respect of units by the horizontal composition simply

by using the exchange identity above with units in the lower two 2–cells. Of course the central leg version of the composition can be directly verified to be functorial. The roundabout route is nice since it covers lots of pasting diagrams, and is a good model for future such verifications.

The unit for composing $\mathcal{V}$–modifications along a $\mathcal{V}$–functor is the identity $\mathcal{V}$–modification $\mathbf{1}_{1_T}$ where $1_T : T \to T$ is the identity $\mathcal{V}$–2–natural transformation for a $\mathcal{V}$–2–functor T. Since $(1_T)_U = \mathcal{J}_{TU}$ then $(1_T)_U(0) = \mathcal{J}_{TU}(0) = 1_{TU}$. Thus $((\mathbf{1}_{1_T})_U)_0 = j_{1_{TU}} = \mathcal{J}_{TU_{00}}$. That $\mathbf{1}_{1_T}$ is a 2–sided unit for the composition is seen by the unit axioms of strict $\mathcal{V}$–2–category.

Proof (Part 2.)

In part 2 of the proof we describe how for each triple of $\mathcal{V}$–2–categories we have a 2–functor of two variables that serves to compose morphisms along a common $\mathcal{V}$–2–category as in the following picture.
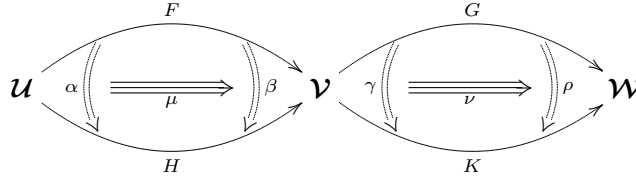
At each stage of description we also need to check that the composition along a common $\mathcal{V}$–2–category is associative and respects all units, as well as making sure that for morphisms the composition is functorial. This latter property always exhibits itself as an exchange identity.

Composition of $\mathcal{V}$–2–functors is just composition of the object functions and composition of the hom–category $\mathcal{V}$–functors, with appropriate subscripts. Thus $(ST)_{UU'}(f) = S_{TUTU'}(T_{UU'}(f))$. Then it is straightforward to verify that the axioms are obeyed, as in

$$(ST)_{U'U''}(f)(ST)_{UU'}(g)$$

$$= S_{TU'TU''}(T_{U'U''}(f))S_{TUTU'}(T_{UU'}(g))$$

$$= S_{TUTU''}(T_{U'U''}(f)T_{UU'}(g))$$

$$= S_{TUTU''}(T_{UU''}(fg))$$

$$= (ST)_{UU''}(fg).$$

That this composition is associative follows from the associativity of composition of the underlying functions and $\mathcal{V}$–functors. The 2–sided identity for this composition $1_{\mathcal{U}}$ is made of the identity function (on objects) and identity $\mathcal{V}$–functors (for hom–categories.)

Next we define the composition of $\mathcal{V}$–2–natural transformations along a $\mathcal{V}$–2–category. This is accomplished by first describing the trivial cases–whiskering a $\mathcal{V}$–2–functor to a $\mathcal{V}$–2–natural transformation along a $\mathcal{V}$–2–category. (By proceeding in terms of whiskers I get the opportunity to both discuss all the

possible pasting in the 3–category and to exhibit the sub–2–categories implicit in its structure.)

The first picture shows a 1-cell ($\mathcal{V}$–2–functor) following a 2-cell ($\mathcal{V}$–2–natural transformation). These are composed to form a new 2-cell as follows
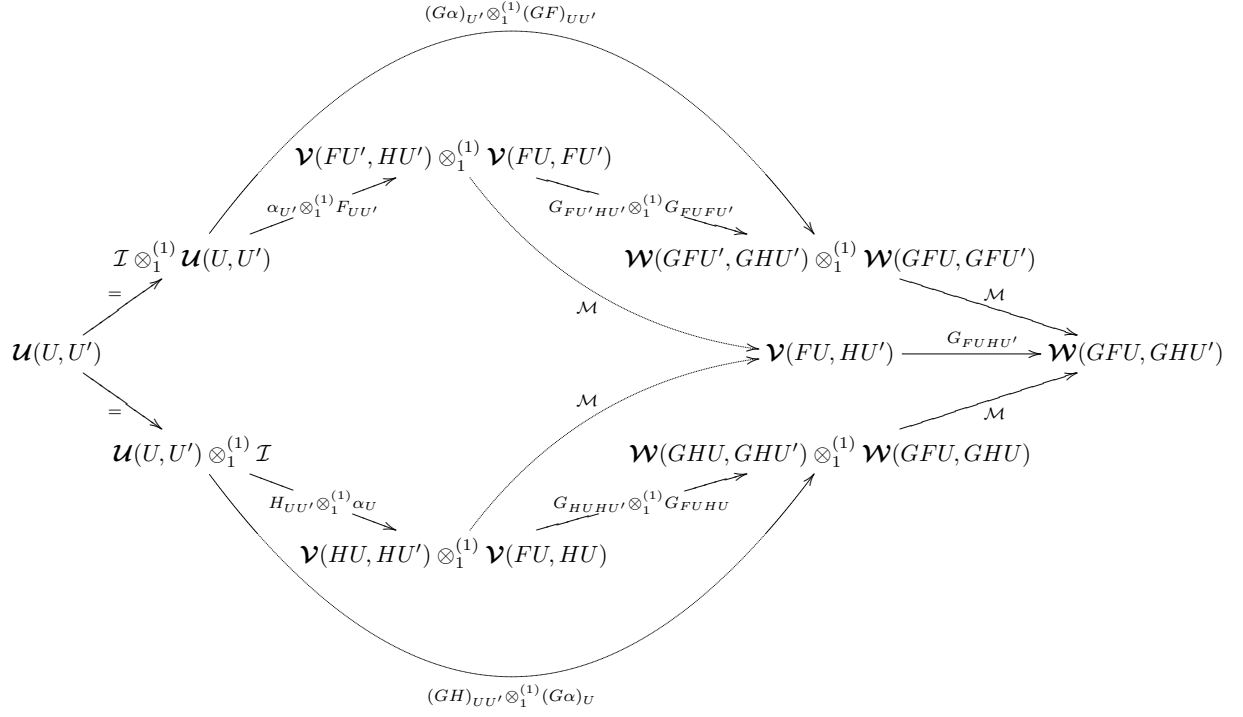

is composed to become

where $G\alpha$ has components given by

$$(G\alpha)_U \;=\; \begin{array}{c} \mathcal{I} \\ \Big\downarrow{\scriptstyle \alpha_U} \\ \mathcal{V}(FU, HU) \\ \Big\downarrow{\scriptstyle G_{FU,HU}} \\ \mathcal{W}(GFU, GHU) \end{array}$$

Notice that by definition and axioms of enrichment, letting $\alpha_U(0) = q$, we have $(G\alpha)_U(0) = Gq$ and $(G\alpha)_{U_{00}} = j_{Gq}$.

This whiskering gives a valid $\mathcal{V}$–2–natural transformation by the following commuting diagram.

The central region expresses the $\mathcal{V}$–2–naturality of α. The two rightmost regions commute by the definition of $\mathcal{V}$–2–functor.

The second picture shows a 2-cell following a 1-cell. These are composed as follows
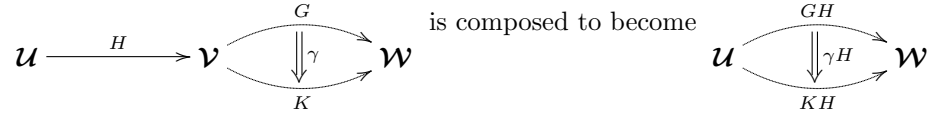
where γH has components given by $(\gamma H)_U = \gamma_{HU}$.

This whiskering gives a valid $\mathcal{V}$–2–natural transformation by the following commuting diagram.

$$(\gamma H)_{U'} \otimes_1^{(1)} (GH)_{UU'}$$

$$\mathcal{I} \otimes_1^{(1)} \boldsymbol{\mathcal{V}}(HU, HU')$$

$$1 \otimes_1^{(1)} H_{UU'}$$

$$\gamma_{HU'} \otimes_1^{(1)} G_{HU\,HU'}$$

$$\mathcal{I} \otimes_1^{(1)} \boldsymbol{\mathcal{U}}(U, U')$$

$$\boldsymbol{\mathcal{W}}(GHU', KHU') \otimes_1^{(1)} \boldsymbol{\mathcal{W}}(GHU, GHU')$$

$$=$$

$$\mathcal{M}$$

$$\boldsymbol{\mathcal{U}}(U, U') \xrightarrow{H_{UU'}} \boldsymbol{\mathcal{V}}(HU, HU')$$

$$\boldsymbol{\mathcal{W}}(GHU, KHU')$$

$$=$$

$$=$$

$$\mathcal{M}$$

$$\boldsymbol{\mathcal{U}}(U, U') \otimes_1^{(1)} \mathcal{I}$$

$$\boldsymbol{\mathcal{W}}(KHU, KHU') \otimes_1^{(1)} \boldsymbol{\mathcal{W}}(GHU, KHU)$$

$$=$$

$$=$$

$$H_{UU'} \otimes_1^{(1)} 1$$

$$K_{HU\,HU'} \otimes_1^{(1)} \gamma_{HU}$$

$$\boldsymbol{\mathcal{V}}(HU, HU') \otimes_1^{(1)} \mathcal{I}$$

$$(KH)_{UU'} \otimes_1^{(1)} (\gamma H)_U$$

The central region expresses the $\mathcal{V}$–2–naturality of γ.

To show the exchange identity here we proceed by checking the usual agreement and functoriality of partials. First I will check that the partial functors described by whiskering are indeed functorial. These proofs continue to parallel the lower dimensional case. First we check that the right whiskering distributes, i.e. that $(Z\alpha) * (Z\beta) = Z(\alpha * \beta)$ as in the following picture. (Recall that "$*$" denotes the composition along $\mathcal{V}$–2–functors as in the first part of the proof.)

The two sides of the proposed equality form the legs of the following diagram, which commutes due to the definition of the $\mathcal{V}$–2–functoriality of Z:

$$\mathcal{I} = \mathcal{I} \otimes_1^{(1)} \mathcal{I}$$

$$\Big\downarrow \alpha_U \otimes_1^{(1)} \beta_U$$

$$\boldsymbol{\mathcal{V}}(SU, RU) \otimes_1^{(1)} \boldsymbol{\mathcal{V}}(TU, SU)$$

$$\swarrow \mathcal{M} \qquad\qquad \searrow Z \otimes_1^{(1)} Z$$

$$\boldsymbol{\mathcal{V}}(TU, RU) \qquad\qquad \boldsymbol{\mathcal{W}}(ZSU, ZRU) \otimes_1^{(1)} \boldsymbol{\mathcal{V}}(ZTU, ZSU)$$

$$\searrow Z \qquad\qquad \swarrow \mathcal{M}$$

$$\boldsymbol{\mathcal{W}}(ZTU, ZRU)$$

For the same requirement on the other partial functor we have the picture

$$\mathcal{U} \xrightarrow{\;\;T\;\;} \mathcal{V} \; \substack{\gamma \\ \Downarrow \\ \Downarrow \\ \delta} \; \mathcal{W}$$

From the definitions is it immediate that $((\delta * \gamma)T)_U = (\delta * \gamma)_{TU} = (\delta T * \gamma T)_U$.

Now we can compose $\mathcal{V}$–2–natural transformations along a $\mathcal{V}$–2–category, as in the following picture.

$$\mathcal{U} \; \substack{F \\ \Downarrow \alpha \\ H} \; \mathcal{V} \; \substack{G \\ \Downarrow \gamma \\ K} \; \mathcal{W}$$

As usual there are two ways to do so that need to be reconciled. They both consist of defining the composition along the $\mathcal{V}$–2–category in terms of a composition along a common $\mathcal{V}$–2–functor as in part 1 of the proof. Thus since the whiskered pieces are valid $\mathcal{V}$–2–natural transformations, by a previous diagram their composition will be as well. The first way of composing is given by:

$$\gamma\alpha \;=\; \mathcal{I} = \mathcal{I} \otimes_1^{(1)} \mathcal{I}$$

$$\Big\downarrow (\gamma H)_U \otimes_1^{(1)} (G\alpha)_U$$

$$\boldsymbol{\mathcal{W}}(GHU, KHU) \otimes_1^{(1)} \boldsymbol{\mathcal{W}}(GFU, GHU)$$

$$\Big\downarrow \mathcal{M}$$

$$\boldsymbol{\mathcal{W}}(GFU, KHU)$$

The second is given by

$$\gamma\alpha \;=\; \begin{array}{c} \mathcal{I} = \mathcal{I} \otimes_1^{(1)} \mathcal{I} \\ \big\downarrow {\scriptstyle (K\alpha)_U \otimes_1^{(1)} (\gamma F)_U} \\ \mathcal{W}(KFU, KHU) \otimes_1^{(1)} \mathcal{W}(GFU, KFU) \\ \big\downarrow {\scriptstyle \mathcal{M}} \\ \mathcal{W}(GFU, KHU) \end{array}$$

Letting $(\gamma F)_U(0) = \hat{q}$ and $(\gamma H)_U(0) = \hat{q}'$ and recalling that $(G\alpha)_U(0) = Gq$ we have $(\gamma\alpha)_U(0) = \hat{q}'Gq = Kq\hat{q}$ and by $\mathcal{V}$–functoriality of $\mathcal{M}$ that $(\gamma\alpha)_{U_{00}} = j_{\hat{q}'Gq}$.

That the two ways of composing are actually the same is based on the $\mathcal{V}$–2–naturality of γ, the definition of which makes up the central region of the following commuting diagram. The other regions commute trivially.

Now whiskering a 1–cell Q on the right (or left) of a 2–cell $\alpha : T \to S$ should be the same as horizontally composing $\mathbf{1}_Q$ on the respective side of α. Pictorially for the right-hand whiskering:

To see this equality we need check only one way of composing $\mathbf{1}_Q\alpha$ since we have shown it to be well defined – i.e. we check that $Q\alpha = \mathbf{1}_Q\alpha = Q\alpha * \mathbf{1}_Q T$. This is true immediately from the relationship of $\mathcal{M}$ and $\mathcal{J}$. Now pictorially for the left-hand whiskering:

That $\alpha P = \alpha \mathbf{1}_P = S\mathbf{1}_P * \alpha P$ also is shown by using the relationship of $\mathcal{M}$ and $\mathcal{J}$ and by the $\mathcal{V}$–2–functoriality of S.

Associativity of this composition follows from the associativity of composing $\mathcal{V}$–2–natural transformations along a $\mathcal{V}$–2–functor. It also requires the functoriality of the partial functors. In the following picture

we have

$$\beta(\gamma\alpha) = Q(K\alpha * \gamma F) * \beta GF$$
$$= (QK\alpha * Q\gamma F) * \beta GF$$
$$= QK\alpha * (Q\gamma F * \beta GF)$$
$$= QK\alpha * (Q\gamma * \beta G)F = (\beta\gamma)\alpha$$

where the assumed associativities of whiskers are easily verified.

The unit for composing $\mathcal{V}$–2–natural transformations along a $\mathcal{V}$–2–category is the identity $\mathcal{V}$–2–natural transformation $\mathbf{1}_{1_{\mathcal{U}}}$ where $1_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}$ is the identity $\mathcal{V}$–2–functor. Note that $(\mathbf{1}_{1_{\mathcal{U}}})_U(0) = \mathcal{J}_{1_{\mathcal{U}}U}(0) = \mathcal{J}_U(0) = 1_U$ and that $(\mathbf{1}_{1_{\mathcal{U}}})_{U_{00}} = \mathcal{J}_{U_{00}} = j_{1_U}$. Since the composition is based on that of $\mathcal{V}$–2–natural transformations along a $\mathcal{V}$–2–functor, to see that $\mathbf{1}_{1_{\mathcal{U}}}$ is a 2–sided unit all we need to check is that for any $\alpha : T \to S : \mathcal{U} \to \mathcal{W}$ we have that $1_{\mathcal{W}}\alpha = \alpha = \alpha 1_{\mathcal{U}}$. This is clear from the definitions of whiskering above.

Next we consider compositions involving $\mathcal{V}$–modifications along a $\mathcal{V}$–2–category. I start by defining whiskering of $\mathcal{V}$–2–functors and then use that definition to define whiskering of $\mathcal{V}$–2–natural transformations. First the right whiskering of a $\mathcal{V}$–2–functor onto a $\mathcal{V}$–modification as in the picture:

where:

$$(K\mu)_U \;=\; \begin{array}{c} I \\ \downarrow{\scriptstyle \mu_U} \\ \boldsymbol{\mathcal{V}}(FU, HU)(q, \check{q}) \\ \downarrow{\scriptstyle K_{FU,HU}} \\ \boldsymbol{\mathcal{W}}(KFU, KHU)(Kq, K\check{q}) \end{array}$$

Where we let $\alpha_U(0) = q$ and $\beta_U(0) = \check{q}$. That this forms a valid $\mathcal{V}$–modification can be seen upon inspecting the following diagram. Its commutativity relies on the fact that μ is a $\mathcal{V}$–modification and on the $\mathcal{V}$–2–functoriality of K.

Secondly the left whiskering of a $\mathcal{V}$–2–functor onto a $\mathcal{V}$–modification as in the picture:

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is given by $(\nu F)_U = \nu_{FU}$.

This one is a valid $\mathcal{V}$–modification because of the following diagram, where we use the fact that ν is a $\mathcal{V}$–modification.

$$\boldsymbol{\mathcal{W}}(GFU', KFU')(\hat{q}', \overline{q}') \otimes_2 \boldsymbol{\mathcal{W}}(GFU, GFU)(GFf, GFg)$$

$$(\nu F)'_U \otimes_2 GF \qquad \nu_{FU'} \otimes_2 G_{FUFU'_{FfFg}} \qquad \mathcal{M}$$

$$I \otimes_2 \boldsymbol{\mathcal{U}}(U,U')(f,g) \xrightarrow{1 \otimes_2 F_{UU'_{fg}}} I \otimes_2 \boldsymbol{\mathcal{V}}(FU,FU')(Ff,Fg) \qquad \boldsymbol{\mathcal{W}}(GFU, KFU')(\hat{q}'GFf, \overline{q}'GFg)$$

$$= \qquad =$$

$$\boldsymbol{\mathcal{U}}(U,U')(f,g) \xrightarrow{F_{UU'_{fg}}} \boldsymbol{\mathcal{V}}(FU,FU')(Ff,Fg)$$

$$= \qquad =$$

$$\boldsymbol{\mathcal{U}}(U,U')(f,g) \otimes_2 I \xrightarrow{F_{UU'_{fg}} \otimes_2 1} \boldsymbol{\mathcal{V}}(FU,FU')(Ff,Fg) \otimes_2 I \qquad \boldsymbol{\mathcal{W}}(GFU, KFU)(KFf\hat{q}, KFg\overline{q})$$

$$KF \otimes_2 (\nu F)_U \qquad K_{FUFU'_{FfFg}} \otimes_2 \nu_{FU} \qquad \mathcal{M}$$

$$\boldsymbol{\mathcal{W}}(KFU, KFU')(KFf, KFg) \otimes_2 \boldsymbol{\mathcal{W}}(GFU, KFU)(\hat{q}, \overline{q})$$

The functoriality of these partials is shown just as for the whiskering of $\mathcal{V}$–2–functors onto $\mathcal{V}$–2–natural transformations. Consider a $\mathcal{V}$–modification $\xi : \phi \to \psi : T \to F : \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{V}}$. For right whiskering we have that $K(\mu * \xi) = K\mu * K\xi$ by the $\mathcal{V}$–2–functoriality of K. For a $\mathcal{V}$–2–functor $S : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{U}}$ we have $(\mu * \xi)S = \mu S * \xi S$ since $((\mu * \xi)S)_X = (\mu * \xi)_{SX} = (\mu S * \xi S)_X$.

In the next step we basically see the generalizations of these last two compositions. Next I define the right whiskering of a $\mathcal{V}$–2–natural transformation onto a $\mathcal{V}$–modification as in the picture:

The $\mathcal{V}$–modification $\rho\mu : \rho\alpha \to \rho\beta$ can be defined in two ways. Let $\alpha_U(0) = q$, $\beta_U(0) = \check{q}$, $\rho_{FU}(0) = \overline{q}$, $\beta_U(0) = \check{q}'$ and $\rho_{HU}(0) = \overline{q}'$. The first way of

composing is given by:

$$\rho\mu_U \;=\; \qquad\qquad I = I \otimes_2 I$$

$$\downarrow {\scriptstyle (K\mu)_U \otimes_2 (\rho F)_{U_{00}}}$$

$$\boldsymbol{\mathcal{W}}(KFU, KHU)(Kq, K\check{q}) \otimes_2 \boldsymbol{\mathcal{W}}(GFU, KFU)(\overline{q}, \overline{q})$$

$$\downarrow {\scriptstyle \mathcal{M}}$$

$$\boldsymbol{\mathcal{W}}(GFU, KHU)(Kq\overline{q}, K\check{q}\overline{q})$$

The second is given by

$$\rho\mu_U \;=\; \qquad\qquad I = I \otimes_2 I$$

$$\downarrow {\scriptstyle (\rho H)_{U_{00}} \otimes_2 (G\mu)_U}$$

$$\boldsymbol{\mathcal{W}}(GHU, KHU)(\overline{q'}, \overline{q'}) \otimes_2 \boldsymbol{\mathcal{W}}(GFU, GHU)(Gq, G\check{q})$$

$$\downarrow {\scriptstyle \mathcal{M}}$$

$$\boldsymbol{\mathcal{W}}(GFU, KHU)(\overline{q'}Gq, \overline{q'}G\check{q})$$

That the two ways agree is given by the following commuting diagram, which depends on the fact that ρ is a $\mathcal{V}$–2–natural transformation.

That this composition yields a $\mathcal{V}$–modification is easily seen when we note that it is by definition the same as composing certain $\mathcal{V}$–modifications along a common $\mathcal{V}$–2–functor. For example the above composition is of the $\mathcal{V}$–modifications $K\mu$ and $1_{\rho F}$ along the $\mathcal{V}$–2–functor KF.

Now we can define the left whiskering of a $\mathcal{V}$–2–natural transformation onto a $\mathcal{V}$–modification as in the picture:
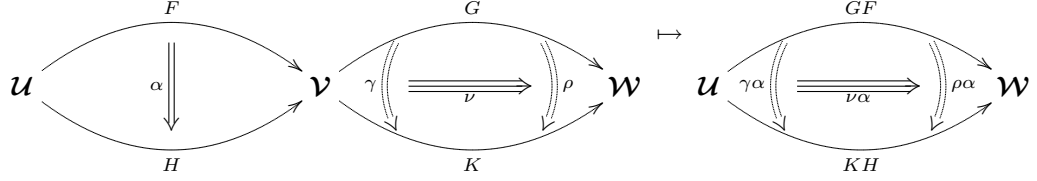
The $\mathcal{V}$–modification $\nu\alpha : \gamma\alpha \to \rho\alpha$ can be defined in two ways. Let $\alpha_U(0) = q$, $\gamma_{FU}(0) = \hat{q}$, $\rho_{FU}(0) = \overline{q}$, $\gamma_{HU}(0) = \hat{q'}$ and $\rho_{HU}(0) = \overline{q'}$. The first way of composing is given by:

$$\nu\alpha_U \;=\; I = I \otimes_2 I$$
$$\downarrow {\scriptstyle (\nu H)_U \otimes_2 (G\alpha)_{U_{00}}}$$
$$\boldsymbol{\mathcal{W}}(GHU, KHU)(\hat{q'}, \overline{q'}) \otimes_2 \boldsymbol{\mathcal{W}}(GFU, GHU)(Gq, Gq)$$
$$\downarrow {\scriptstyle \mathcal{M}}$$
$$\boldsymbol{\mathcal{W}}(GFU, KHU)(\hat{q'}Gq, \overline{q'}Gq)$$

The second is given by

$$\nu\alpha_U \;=\; I = I \otimes_2 I$$
$$\downarrow {\scriptstyle (K\alpha)_{U_{00}} \otimes_2 (\nu F)_U}$$
$$\boldsymbol{\mathcal{W}}(KFU, KHU)(Kq, Kq) \otimes_2 \boldsymbol{\mathcal{W}}(GFU, KFU)(\hat{q}, \overline{q})$$
$$\downarrow {\scriptstyle \mathcal{M}}$$
$$\boldsymbol{\mathcal{W}}(GFU, KHU)(Kq\hat{q}, Kq\overline{q})$$

That the two ways agree is given by the following commuting diagram, which depends on the fact that ν is a $\mathcal{V}$–modification.

$$\mathcal{W}(GHU, KHU)(\hat{q}', \overline{q}') \otimes_2 \mathcal{W}(GFU, GHU)(Gq, Gq)$$

$$(\nu H)_U \otimes_2 (G\alpha)_U \qquad \nu_{HU} \otimes_2 G_{FUHU_{qq}} \qquad \Big\downarrow \mathcal{M}$$

$$I \otimes_2 I \xrightarrow{\ 1 \otimes_2 \alpha_{U_{00}}\ } I \otimes_2 \mathcal{V}(FU, HU)(q,q) \qquad \mathcal{W}(GFU, KHU)(\hat{q}'Gq, \overline{q}'Gq)$$

$$=\qquad\qquad =$$

$$I \xrightarrow{\ \alpha_{U_{00}}\ } \mathcal{V}(FU, HU)(q,q)$$

$$=\qquad\qquad =$$

$$I \otimes_2 I \xrightarrow{\ \alpha_{U_{00}} \otimes_2 1\ } \mathcal{V}(FU, HU)(q,q) \otimes_2 I \qquad \mathcal{W}(GFU, KHU)(Kq\hat{q}, Kq\overline{q})$$

$$(K\alpha)_U \otimes_2 (\nu F)_U \qquad K_{FUHU_{qq}} \otimes_2 \nu_{FU} \qquad \Big\uparrow \mathcal{M}$$

$$\mathcal{W}(KFU, KHU)(Kq, Kq) \otimes_2 \mathcal{W}(GFU, KFU)(\hat{q}, \overline{q})$$

Again this composition yields a $\mathcal{V}$–modification since it is by definition the same as composing certain $\mathcal{V}$–modifications along a common $\mathcal{V}$–2–functor.

Necessary for the functoriality of the partials given by the above left and right whiskering is that we have $\rho(\omega \circ \mu) = \rho\omega \circ \rho\mu$ and $(tau \circ \nu)\alpha = (\tau\alpha \circ \nu\alpha)$ as in the following pictures:

These requirements are met since the exteriors of the following two diagrams commute, respectively. Here let $\delta_U(0) = \hat{\hat{q}}$ and $\tau_U(0) = \overline{\overline{q}}$.

$$\mathcal{W}(GFU,KHU)(K\bar{q}\hat{q},K\hat{\bar{q}q}) \otimes_1 \mathcal{W}(GFU,KHU)(K\hat{q}\bar{q},K\bar{q}\hat{q})$$

$$(\mathcal{W}(KFU,KHU)(K\hat{q},K\hat{\bar{q}}) \otimes_2 \mathcal{W}(GFU,KFU)(\bar{q},\bar{q})) \otimes_1 (\mathcal{W}(KFU,KHU)(Kq,K\hat{q}) \otimes_2 (\mathcal{W}(GFU,KFU)(\bar{q},\bar{q}) \otimes_1 \mathcal{W}(GFU,KFU)(\bar{q},\bar{q}))$$

$$(\mathcal{W}(KFU,KHU)(K\hat{q},K\hat{\bar{q}}) \otimes_1 \mathcal{W}(KFU,KHU)(Kq,K\hat{q})) \otimes_2 (\mathcal{W}(GFU,KFU)(\bar{q},\bar{q}) \otimes_1 \mathcal{W}(GFU,KFU)(\bar{q},\bar{q}))$$

$$I = I \otimes_1 I = (I \otimes_2 I) \otimes_1 (I \otimes_2 I) \xrightarrow{\ \eta\ } (I \otimes_1 I) \otimes_2 (I \otimes_1 I) = I \otimes_2 I \xrightarrow{\ K(\omega\mu)\otimes_2\rho F\ } \mathcal{W}(KFU,KHU)(Kq,K\hat{q}) \otimes_2 \mathcal{W}(GFU,KFU)(\bar{q},\bar{q}) \xrightarrow{\ \mathcal{M}\ } \mathcal{W}(GFU,KHU)(Kq\bar{q},K\hat{\bar{q}q})$$

Arrow labels: $\mathcal{M}\otimes_1\mathcal{M}$, $\mathcal{M}\otimes_2\mathcal{M}$, η, $\mathcal{M}$, $(K\omega\otimes_2\rho F)\otimes_1(K\mu\otimes_2\rho F)$, $(K\omega\otimes_1 K\mu)\otimes_2(\rho F\otimes_1\rho F)$

$$\mathcal{W}(GFU,KHU)(Kq\hat{q},Kq\hat{q}) \otimes_1 \mathcal{W}(GFU,KHU)(Kq\hat{q},Kq\hat{q})$$

$$(\mathcal{W}(KFU,KHU)(Kq,Kq) \otimes_2 \mathcal{W}(GFU,KFU)(\hat{q},\hat{q})) \otimes_1 (\mathcal{W}(KFU,KHU)(Kq,Kq) \otimes_2 \mathcal{W}(GFU,KFU)(\bar{q},\bar{q}))$$

$$(\mathcal{W}(KFU,KHU)(Kq,Kq) \otimes_1 \mathcal{W}(KFU,KHU)(Kq,Kq)) \otimes_2 (\mathcal{W}(GFU,KFU)(\bar{q},\bar{q}) \otimes_1 \mathcal{W}(GFU,KFU)(\hat{q},\hat{q}))$$

$$I = I \otimes_1 I = (I \otimes_2 I) \otimes_1 (I \otimes_2 I) \xrightarrow{\ \eta\ } (I \otimes_1 I) \otimes_2 (I \otimes_1 I) = I \otimes_2 I \xrightarrow{\ K\alpha\otimes_2(\tau\circ\nu)F\ } \mathcal{W}(KFU,KHU)(Kq,Kq) \otimes_2 \mathcal{W}(GFU,KFU)(\hat{q},\hat{q}) \xrightarrow{\ \mathcal{M}\ } \mathcal{W}(GFU,KHU)(Kq\hat{q},Kq\hat{q})$$

Arrow labels: $\mathcal{M}\otimes_2\mathcal{M}$, η, $\mathcal{M}$, $\mathcal{M}\otimes_1\mathcal{M}$, $(K\alpha\otimes_1 K\alpha)\otimes_2(\tau F\otimes_1\nu F)$, $(K\alpha\otimes_2\tau F)\otimes_1(K\alpha\otimes_2\nu F)$

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It is straightforward to check that if an identity $\mathcal{V}$–2–natural transformation for a given $\mathcal{V}$–2–functor is whiskered onto the left or right of a $\mathcal{V}$–modification the definitions give exactly the respective whiskering of the $\mathcal{V}$–2–functor itself. Thus the following definition of the horizontal composition of $\mathcal{V}$–modifications along a $\mathcal{V}$–2–category, given in terms of composing along a common $\mathcal{V}$–2–natural transformation could be written less generally but equivalently in terms of composing along a common $\mathcal{V}$–2–functor. Either way the result is a valid $\mathcal{V}$–modification based on an earlier proof. The equivalence will actually be a corollary of the proof of the well defined nature of the composition. Now we are considering the full picture:
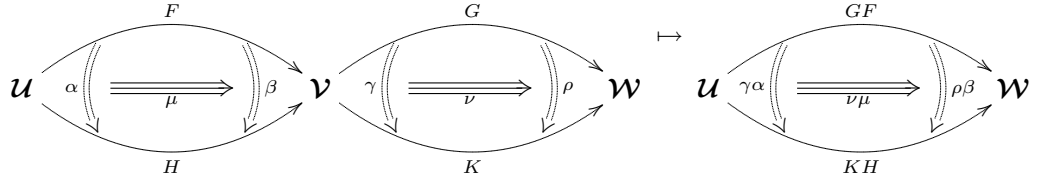
and the two ways of defining $\nu\mu$ in terms of composing along a common $\mathcal{V}$–2–natural transformation are as follow.

The first is:
$$(\nu\mu)_U = (\rho\mu \circ \nu\alpha)_U \ =$$

$$I = I \otimes_1 I = (I \otimes_2 I) \otimes_1 (I \otimes_2 I)$$
$$\downarrow {\scriptstyle ((K\mu)_U \otimes_2 (\rho F)_{U_{00}}) \otimes_1 ((K\alpha)_{U_{00}} \otimes_2 (\nu F)_U)}$$
$$(\boldsymbol{\mathcal{W}}(KFU, KHU)(Kq, K\check{q}) \otimes_2 \boldsymbol{\mathcal{W}}(GFU, KFU)(\overline{q}, \overline{q}) \otimes_1 (\boldsymbol{\mathcal{W}}(KFU, KHU)(Kq, Kq) \otimes_2 \boldsymbol{\mathcal{W}}(GFU, KFU)(\hat{q}, \overline{q}))$$
$$\downarrow {\scriptstyle \mathcal{M} \otimes_1 \mathcal{M}}$$
$$\boldsymbol{\mathcal{W}}(GFU, KHU)(Kq\overline{q}, K\check{q}\overline{q}) \otimes_1 \boldsymbol{\mathcal{W}}(GFU, KHU)(Kq\hat{q}, Kq\overline{q})$$
$$\downarrow {\scriptstyle M}$$
$$\boldsymbol{\mathcal{W}}(GFU, KHU)(Kq\hat{q}, K\check{q}\overline{q})$$

and the second:
$$(\nu\mu)_U = (\nu\beta \circ \gamma\mu)_U \ =$$

$$I = I \otimes_1 I = (I \otimes_2 I) \otimes_1 (I \otimes_2 I)$$

$$\downarrow {\scriptstyle ((\nu H)_U \otimes_2 (G\beta)_{U_{00}}) \otimes_1 ((\gamma H)_{U_{00}} \otimes_2 (G\mu)_U)}$$

$$(\boldsymbol{\mathcal{W}}(GHU, KHU)(\hat{q}', \overline{q}') \otimes_2 \boldsymbol{\mathcal{W}}(GFU, GHU)(G\breve{q}, G\breve{q})) \otimes_1 (\boldsymbol{\mathcal{W}}(GHU, KHU)(\hat{q}', \hat{q}') \otimes_2 \boldsymbol{\mathcal{W}}(GFU, GHU)(Gq, G\breve{q}))$$

$$\downarrow {\scriptstyle \mathcal{M} \otimes_1 \mathcal{M}}$$

$$\boldsymbol{\mathcal{W}}(GFU, KHU)(\hat{q}'G\breve{q}, \overline{q}'G\breve{q}) \otimes_1 \boldsymbol{\mathcal{W}}(GFU, KHU)(\hat{q}'Gq, \hat{q}'G\breve{q})$$

$$\downarrow {\scriptstyle M}$$

$$\boldsymbol{\mathcal{W}}(GFU, KHU)(\hat{q}'Gq, \overline{q}'G\breve{q})$$

Note that in both of the preceding two definitions we have made two choices between equivalent ways of representing component $\mathcal{V}$–modifications. The preceding two definitions are equivalent based on the following commutative diagram.

$(\boldsymbol{\mathcal{W}}(GHU,KHU)(\hat{q'},\overline{q'}) \otimes_2 \boldsymbol{\mathcal{W}}(GFU,GHU)(G\check{q},G\check{q})) \otimes_1 (\boldsymbol{\mathcal{W}}(GHU,KHU)(\hat{q'},\hat{q'}) \otimes_2 \boldsymbol{\mathcal{W}}(GFU,GHU)(Gq,G\check{q}))$

$((\nu H)_U \otimes_2 (G\beta)_{U_{00}}) \otimes_1 ((\gamma H)_{U_{00}} \otimes_2 (G\mu)_U)$

$(I \otimes_2 I) \otimes_1 (I \otimes_2 I)$

η

$\mathcal{M} \otimes_1 \mathcal{M}$

$=$

$(I \otimes_1 I) \otimes_2 (I \otimes_1 I)$

η

$=$

$M \otimes_2 M$

$I \otimes_1 I$

$(\nu H)_U \otimes_2 (G\mu)_U$

$\boldsymbol{\mathcal{W}}(GHU,KHU)(\hat{q'},\overline{q'}) \otimes_2 \boldsymbol{\mathcal{W}}(GFU,GHU)(Gq,G\check{q})$

$I \otimes_2 I$

$1 \otimes_2 \mu_U$

$\nu_{HU} \otimes_2 G_{\widehat{FUHU}}$

$\mathcal{M}$

$=$

$I \otimes_2 \boldsymbol{\mathcal{V}}(FU,HU)(q,\check{q})$

$\boldsymbol{\mathcal{W}}(GFU,KHU)(\hat{q'}Gq,\overline{q'}G\check{q})$

$=$

$I \xrightarrow{\mu_U} \boldsymbol{\mathcal{V}}(FU,HU)(q,\check{q})$

M

$=$

$=$

$\boldsymbol{\mathcal{V}}(FU,HU)(q,\check{q}) \otimes_2 I$

$\boldsymbol{\mathcal{W}}(GFU,KHU)(Kq\hat{q},K\check{q}\overline{q})$

$\mu_U \otimes_2 1$

$K_{FUHU} \otimes_2 \nu_{FU}$

$\mathcal{M}$

$I \otimes_2 I$

$(K\mu)_U \otimes_2 (\nu F)_U$

$\boldsymbol{\mathcal{W}}(KFU,KHU)(Kq,K\check{q}) \otimes_2 \boldsymbol{\mathcal{W}}(GFU,KFU)(\hat{q},\overline{q})$

M

$I \otimes_1 I$

$=$

$M \otimes_2 M$

$=$

$(I \otimes_1 I) \otimes_2 (I \otimes_1 I)$

η

$\mathcal{M} \otimes_1 \mathcal{M}$

$(I \otimes_2 I) \otimes_1 (I \otimes_2 I)$

η

$((K\mu)_U \otimes_2 (\rho F)_{U_{00}}) \otimes_1 ((K\alpha)_{U_{00}} \otimes_2 (\nu F)_U)$

$(\boldsymbol{\mathcal{W}}(KFU,KHU)(Kq,K\check{q}) \otimes_2 \boldsymbol{\mathcal{W}}(GFU,KFU)(\overline{q},\overline{q})) \otimes_1 (\boldsymbol{\mathcal{W}}(KFU,KHU)(Kq,Kq) \otimes_2 \boldsymbol{\mathcal{W}}(GFU,KFU)(\hat{q},\overline{q}))$

The exterior commutes since all the interior regions commute. The top and bottom bullets are labeled by the text at the top and bottom of the diagram. The other bullets and unlabeled arrows should be easily filled in, noting that the uppermost and lowest quadrilaterals commute by the naturality of η. The arrows marked with an "=" all occur as copies of I are tensored to the object at the arrow's source. Therefore the western regions with the initial I as a vertex all commute trivially. The large central region expresses the fact that ν is a $\mathcal{V}$–modification. The pentagonal regions on the right commute by the $\mathcal{V}$–functoriality of $\mathcal{M}$. The remaining interior regions commute by definition and by the axioms of a $\mathcal{V}$–category.

The thick arrows in the central portion of the above diagram outline the definition of composing $\mathcal{V}$–modifications along a $\mathcal{V}$–2–category in terms of composing along a common $\mathcal{V}$–2–functor. Thus this diagram also demonstrates that the two ways of doing so are equivalent to each other and to the method which uses composition along a common $\mathcal{V}$–2–natural transformation.

Next we continue checking functoriality of partials. As usual I check the stronger property that the composition defined by those partials gives the whiskering itself as a composition with a unit. First we check that composing in the following two pictures yields the same $\mathcal{V}$–modification.

$$\mathcal{U} \xLeftrightarrow[H]{F} \mathcal{V} \xrightarrow[K]{G} \mathcal{W} \quad = \quad \mathcal{U} \xLeftrightarrow[H]{F} \mathcal{V} \xLeftrightarrow[K]{G} \mathcal{W}$$

Using the definition of the composition of $\mathcal{V}$–modifications in terms of composing along a common $\mathcal{V}$–2–functor it is easy to see that this equality follows from the fact that $(\rho F)_{U_{00}} = (\rho)_{FU_{00}} = j_{(\rho FU(0))} = (1_\rho)_{FU} = (1_\rho F)_U$.

As noted earlier the compositions in the first two of the following pictures yield the same $\mathcal{V}$–modifications as well, due to the fact that $(1_G F)_{U_{00}} = (1_G)_{FU_{00}} = \mathcal{J}_{GFU_{00}}$. Thus by the above equality all three are equivalent:

$$\mathcal{U} \xLeftrightarrow[H]{F} \mathcal{V} \xrightarrow{G} \mathcal{W}$$

$$=$$

$$\mathcal{U} \xLeftrightarrow[H]{F} \mathcal{V} \xLeftrightarrow[G]{G} \mathcal{W}$$

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=

$$\mathcal{U} \xrightarrow[\ H\ ]{\ F\ }\overset{\alpha}{\underset{\beta}{\Downarrow\mu\Downarrow}} \mathcal{V} \xrightarrow[\ G\ ]{\ G\ }\overset{1_G}{\underset{1_G}{\Downarrow 1_{1_G}\Downarrow}} \mathcal{W}$$

On the other side we need to check that the following compositions are equivalent:

$$\mathcal{U} \xrightarrow[\ H\ ]{\ F\ }\Big\Downarrow{\alpha} \mathcal{V} \xrightarrow[\ K\ ]{\ G\ }\overset{\gamma}{\underset{\rho}{\Downarrow\nu\Downarrow}} \mathcal{W} \quad = \quad \mathcal{U} \xrightarrow[\ H\ ]{\ F\ }\overset{\alpha}{\underset{\alpha}{\Downarrow 1_\alpha\Downarrow}} \mathcal{V} \xrightarrow[\ K\ ]{\ G\ }\overset{\gamma}{\underset{\rho}{\Downarrow\nu\Downarrow}} \mathcal{W}$$

This follows from the definition of $\mathcal{V}$–2–functor (relation to unit axiom) since $(G\alpha)_{U_{00}} = j_{G(\alpha_U(0))} = (G1_\alpha)_U$. Furthermore, since $(G1_F)_{U_{00}} = \mathcal{J}_{GFU_{00}}$, we have the equality:

$$\mathcal{U} \xrightarrow{\ F\ } \mathcal{V} \xrightarrow[\ K\ ]{\ G\ }\overset{\gamma}{\underset{\rho}{\Downarrow\nu\Downarrow}} \mathcal{W}$$

=

$$\mathcal{U} \xrightarrow[\ F\ ]{\ F\ }\Big\Downarrow{1_F} \mathcal{V} \xrightarrow[\ K\ ]{\ G\ }\overset{\gamma}{\underset{\rho}{\Downarrow\nu\Downarrow}} \mathcal{W}$$

=

$$\mathcal{U} \xrightarrow[\ F\ ]{\ F\ }\overset{1_F}{\underset{1_F}{\Downarrow 1_{1_F}\Downarrow}} \mathcal{V} \xrightarrow[\ K\ ]{\ G\ }\overset{\gamma}{\underset{\rho}{\Downarrow\nu\Downarrow}} \mathcal{W}$$

Associativity of this composition follows from the associativity of composing $\mathcal{V}$–modifications along a $\mathcal{V}$–2–functor. It also requires the functoriality of the partial functors that describe whiskering $\mathcal{V}$–2–functors onto $\mathcal{V}$–modifications. In the following picture
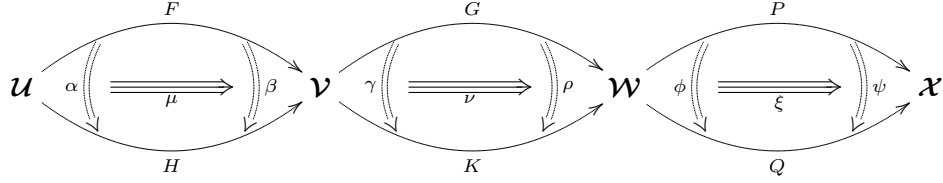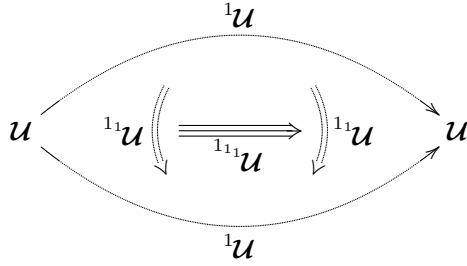
we have

$$\xi(\nu\mu) = Q(K\mu * \nu F) * \xi GF$$
$$= (QK\mu * Q\nu F) * \xi GF$$
$$= QK\mu * (Q\nu F * \xi GF)$$
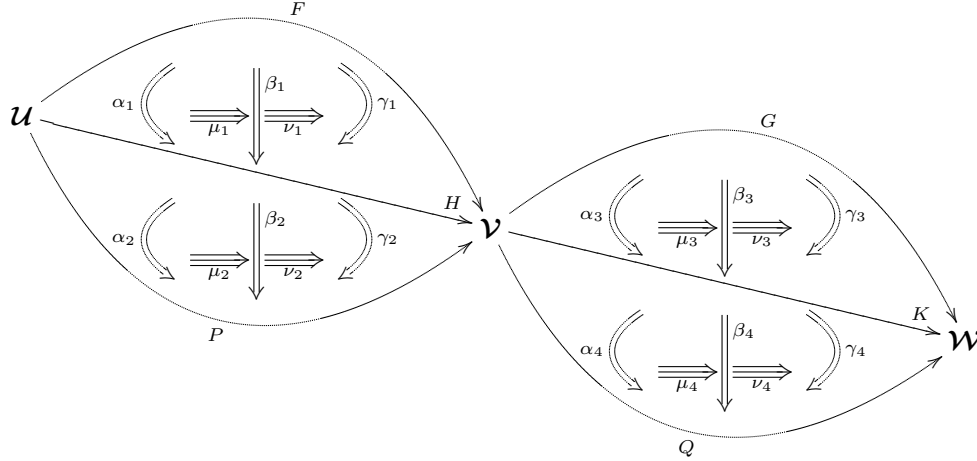$$= QK\mu * (Q\nu * \xi G)F = (\xi\nu)\mu$$

where the assumed associativities of whiskers are easily verified.

Finally the unit for this composition is given by the $\mathcal{V}$–modification $1_{1_{1_{\mathcal{U}}}}$ as in the following picture:

It is straightforward to check that this is a 2–sided unit for the composition of $\mathcal{V}$–modifications along a common $\mathcal{V}$–2–category once we recognize that $(1_{1_{1_{\mathcal{U}}}})_U : I \to \mathcal{U}(U,U)(1_U,1_U)$ is the morphism j_{1_U} in $\mathcal{V}$.

I close with the basic pasting diagram that the above proof has shown to be well-defined. There are 4 exchange identities that this well–definedness depends upon, the requirements for each of which have been met.

Thus we have:

$$(\alpha_4 \alpha_2) * (\alpha_3 \alpha_1) = (\alpha_4 * \alpha_3)(\alpha_2 * \alpha_1)$$

$$(\nu_2 \circ \mu_2) * (\nu_1 \circ \mu_1) = (\nu_2 * \nu_1) \circ (\mu_2 * \mu_1)$$

$$(\nu_3 \nu_1) \circ (\mu_3 \mu_1) = (\nu_3 \circ \mu_3)(\nu_1 \circ \mu_1)$$

$$(\mu_4 \mu_2) * (\mu_3 \mu_1) = (\mu_4 * \mu_3)(\mu_2 * \mu_1).$$

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