ENRICHED 2-NATURAL TRANSFORMATIONS, MODIFICATIONS, AND HIGHER MORPHISMS

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Abstract. We review enriched 2-natural transformations, modifications, and higher morphisms in the context of a symmetric monoidal n-category $\mathcal{V}$. The goal is to discern what sort of algebraic structure these actually comprise.

Contents

1. Introduction

Here we go over the definitions of strict morphisms for strict enriched n-categories. This includes higher enriched natural transformations and modifications. This is a symmetric category based version of the same information found in [Forcey2, 2004]. Thus the proofs there are more general than the omitted proofs here.

2. Categories Enriched over $\mathcal{V}$–Cat

Recall that the unit $\mathcal{V}$–category $\mathcal{I}$ has only one object 0 and $\mathcal{I}(0,0) = I$ the unit in $\mathcal{V}$.

2.1. **Example.** A (small,strict) *$\mathcal{V}$–2–category* $\boldsymbol{\mathcal{U}}$ consists of

(1) A set of objects $|\boldsymbol{\mathcal{U}}|$
(2) For each pair of objects $A, B \in |\boldsymbol{\mathcal{U}}|$ a $\mathcal{V}$–category $\boldsymbol{\mathcal{U}}(A,B)$.
Of course then $\boldsymbol{\mathcal{U}}(A,B)$ consists of a set of objects (which play the role of the 1–cells in a 2–category) and for each pair $f, g \in |\boldsymbol{\mathcal{U}}(A,B)|$ an object $\boldsymbol{\mathcal{U}}(A,B)(f,g) \in \mathcal{V}$ (which plays the role of the hom–set of 2–cells in a 2–category.) Thus the vertical composition morphisms of these hom_2–objects are in $\mathcal{V}$:

$$M_{fgh} : \boldsymbol{\mathcal{U}}(A,B)(g,h) \otimes \boldsymbol{\mathcal{U}}(A,B)(f,g) \to \boldsymbol{\mathcal{U}}(A,B)(f,h)$$

Also, the vertical identity for a 1-cell object $a \in |\boldsymbol{\mathcal{U}}(A,B)|$ is $j_a : I \to \boldsymbol{\mathcal{U}}(A,B)(a,a)$. The associativity and the units of vertical composition are then those given by the respective axioms of enriched categories.

Key words and phrases. enriched categories, n-categories, iterated monoidal categories.
Thanks to XY-pic for the diagrams.

(3) For each triple of objects $A, B, C \in |\boldsymbol{\mathcal{U}}|$ a $\mathcal{V}$–functor

$$\mathcal{M}_{ABC} : \boldsymbol{\mathcal{U}}(B,C) \otimes \boldsymbol{\mathcal{U}}(A,B) \to \boldsymbol{\mathcal{U}}(A,C)$$

Often we repress the subscripts. We denote $\mathcal{M}(h, f)$ as hf.

The family of morphisms indexed by pairs of objects $(g, f), (g', f') \in |\boldsymbol{\mathcal{U}}(B,C) \otimes \boldsymbol{\mathcal{U}}(A,B)|$ furnishes the direct analogue of horizontal composition of 2-cells as can be seen by observing their domain and range in $\mathcal{V}$:

$$\mathcal{M}_{ABC_{(g,f)(g',f')}} : [\boldsymbol{\mathcal{U}}(B,C) \otimes \boldsymbol{\mathcal{U}}(A,B)]((g,f),(g',f')) \to \boldsymbol{\mathcal{U}}(A,C)(gf, g'f')$$

Recall that

$$[\boldsymbol{\mathcal{U}}(B,C) \otimes \boldsymbol{\mathcal{U}}(A,B)]((g,f),(g',f')) = \boldsymbol{\mathcal{U}}(B,C)(g,g') \otimes \boldsymbol{\mathcal{U}}(A,B)(f,f').$$

We can now form the partial functors $\mathcal{M}(h, -) : \boldsymbol{\mathcal{U}}(A,B) \to \boldsymbol{\mathcal{U}}(A,C)$ given by

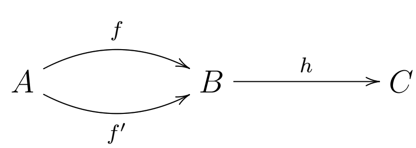
$$\begin{array}{c} \boldsymbol{\mathcal{U}}(A,B) = \mathcal{I} \otimes \boldsymbol{\mathcal{U}}(A,B) \; . \\ \downarrow{\scriptstyle h\otimes 1} \\ \boldsymbol{\mathcal{U}}(B,C) \otimes \boldsymbol{\mathcal{U}}(A,B) \\ \downarrow{\scriptstyle \mathcal{M}} \\ \boldsymbol{\mathcal{U}}(A,C) \end{array}$$

Where h is here seen as the constant functor.

Then $\mathcal{M}(h, -)_{ff'}$ is given by

$$\begin{array}{c} \boldsymbol{\mathcal{U}}(A,B)(f,f') = I \otimes \boldsymbol{\mathcal{U}}(A,B)(f,f') \; . \\ \downarrow{\scriptstyle j_h\otimes 1} \\ \boldsymbol{\mathcal{U}}(B,C)(h,h) \otimes \boldsymbol{\mathcal{U}}(A,B)(f,f') \\ \downarrow{\scriptstyle \mathcal{M}_{(h,f)(h,f')}} \\ \boldsymbol{\mathcal{U}}(A,C)(hf, hf') \end{array}$$

This is the analogue of whiskering on the right. We can heuristically represent the objects of $\boldsymbol{\mathcal{U}}$(A,B) as arrows in a diagram. The diagram for $\mathcal{M}(h, -)_{ff'}$ should be



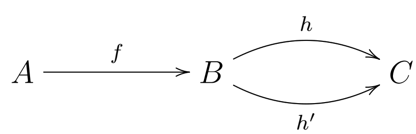
The other partial functors are $\mathcal{M}(-,f) : \boldsymbol{\mathcal{U}}(B,C) \to \boldsymbol{\mathcal{U}}(A,C)$ given by

$$\begin{array}{c} \boldsymbol{\mathcal{U}}(B,C) = \boldsymbol{\mathcal{U}}(B,C) \otimes \mathcal{I} \; . \\ {\scriptstyle 1\otimes f}\big\downarrow \\ \boldsymbol{\mathcal{U}}(B,C) \otimes \boldsymbol{\mathcal{U}}(A,B) \\ {\scriptstyle \mathcal{M}}\big\downarrow \\ \boldsymbol{\mathcal{U}}(A,C) \end{array}$$

Then $\mathcal{M}(-,f)_{hh'}$ is given by

$$\begin{array}{c} \boldsymbol{\mathcal{U}}(B,C)(h,h') = \boldsymbol{\mathcal{U}}(B,C)(h,h') \otimes I \; . \\ {\scriptstyle 1\otimes j_f}\big\downarrow \\ \boldsymbol{\mathcal{U}}(B,C)(h,h') \otimes \boldsymbol{\mathcal{U}}(A,B)(f,f) \\ {\scriptstyle \mathcal{M}_{(h,f)(h',f)}}\big\downarrow \\ \boldsymbol{\mathcal{U}}(A,C)(hf,h'f) \end{array}$$

This is the analogue of whiskering on the left, as in



Notice that given any pair of partial functors, they can be combined to give a unique full functor since $\mathcal{V}$ is symmetric.

(4) For each object $A \in |\boldsymbol{\mathcal{U}}|$ a $\mathcal{V}$–functor

$$\mathcal{J}_A : \mathcal{I} \to \boldsymbol{\mathcal{U}}(A,A)$$

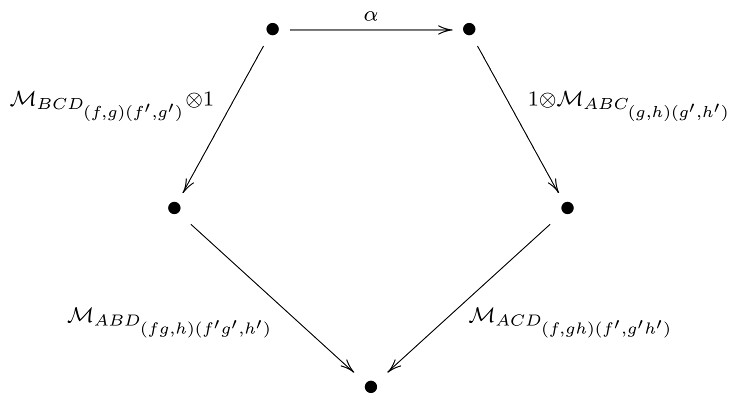
We denote $\mathcal{J}_A(0)$ as 1_A.

(5) (Associativity and unit axioms of a strict $\mathcal{V}$–2–category.) For comparison see [Kelly, 1982] for the analogous axioms in the definition of enriched category. Since now the morphisms are $\mathcal{V}$–functors this amounts to saying that the functors given by the two legs of a diagram are equal. For objects here we then have the equalities $(fg)h = f(gh)$ and $f1_A = f = 1_Bf$

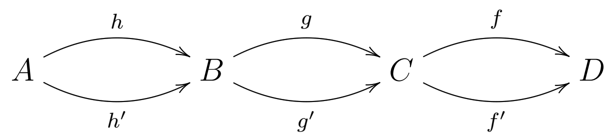
For the hom–object morphisms we have the following family of commuting diagrams for associativity, where the first bullet represents

$$[(\boldsymbol{\mathcal{U}}(C,D) \otimes \boldsymbol{\mathcal{U}}(B,C)) \otimes \boldsymbol{\mathcal{U}}(A,B)](((f,g),h),((f',g'),h'))$$

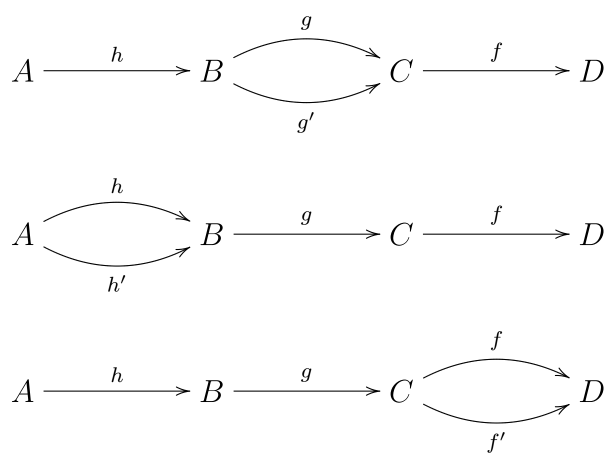
and the reader may fill in the others



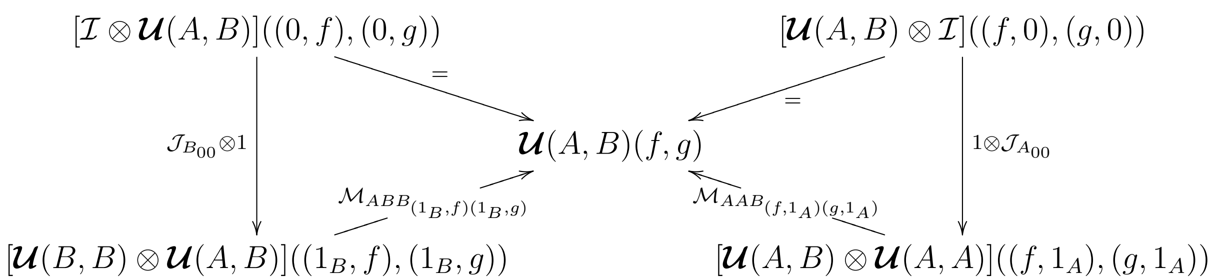
The heuristic diagram for this commutativity is



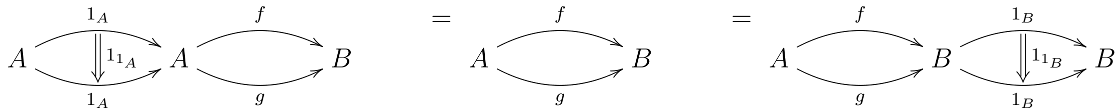
Some special cases in this family of commuting diagrams mentioned in [Lyubashenko, 2003] are those described by the following heuristic diagrams.



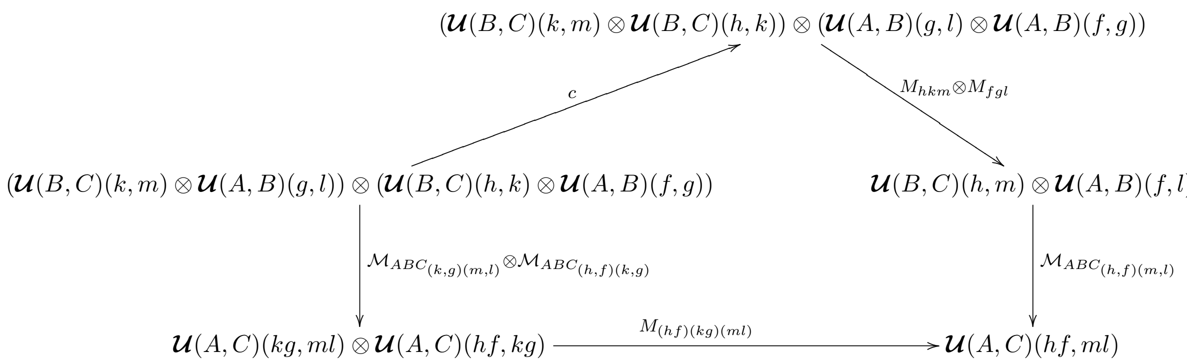
For the unit morphisms we have that the triangles in the following diagram commute.



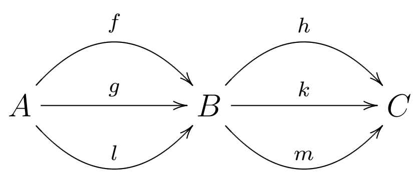
The heuristic diagrams for this commutativity are



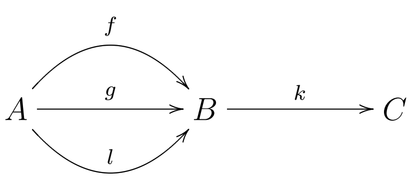
2.2. **Theorem.** *$\mathcal{V}$–functoriality of $\mathcal{M}$ and $\mathcal{J}$: First the $\mathcal{V}$–functoriality of $\mathcal{M}$ implies that the following (expanded) diagram commutes*



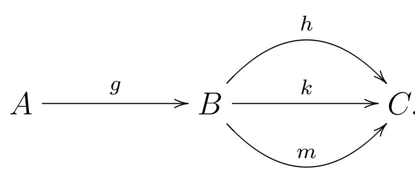
The heuristic diagram is



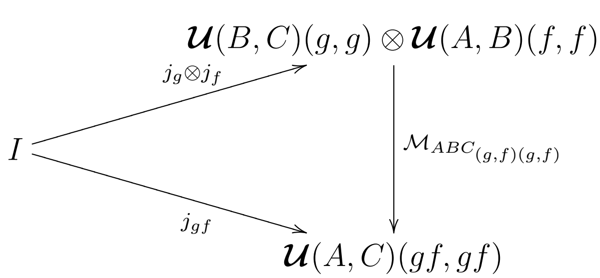
$\mathcal{V}$–functoriality of $\mathcal{M}$ implies $\mathcal{V}$–functoriality of the partial functors $\mathcal{M}(h,-)$. Special cases mentioned in [Lyubashenko, 2003] *include those described by the diagrams*



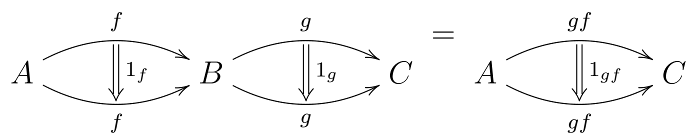
and



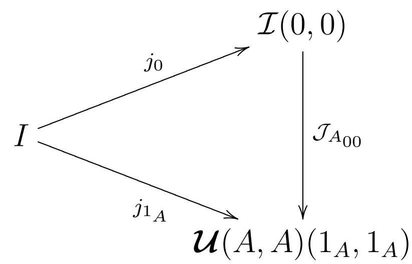
Secondly the $\mathcal{V}$–functoriality of $\mathcal{M}$ implies that the following (expanded) diagram commutes



The heuristic diagram here is



In addition, the $\mathcal{V}$–functoriality of $\mathcal{J}$ implies that the following (expanded) diagram commutes



Which means that

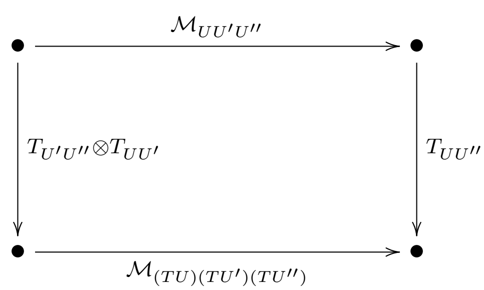
$$\mathcal{J}_{A_{00}} : I \to \boldsymbol{\mathcal{U}}(A, A)(1_A, 1_A) = j_{1_A}.$$

In other words the "horizontal" unit for the object 1_A is the same as the "vertical" unit for 1_A.

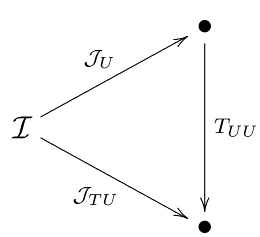
We now describe the (strict) 3–category $\mathcal{V}$–2–Cat (or $\mathcal{V}$–Cat–Cat) whose objects are (strict, small) $\mathcal{V}$–2–categories. We are guided by the definitions of $\mathcal{V}$–functor and $\mathcal{V}$–natural transformation as well as by the definitions of 2–functor, 2–natural transformation, and modification.

2.3**. Definition.** For two $\mathcal{V}$–2–categories $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{W}}$ a *$\mathcal{V}$–2–functor* $T : \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{W}}$ is a function on objects $|\boldsymbol{\mathcal{U}}| \to |\boldsymbol{\mathcal{W}}|$ and a family of $\mathcal{V}$–functors $T_{UU'} : \boldsymbol{\mathcal{U}}(U, U') \to \boldsymbol{\mathcal{W}}(TU, TU')$. These latter obey commutativity of the usual diagrams.

(1) For $U, U', U'' \in |\boldsymbol{\mathcal{U}}|$



(2)



For objects this means that $T_{U'U''}(f)T_{UU'}(g) = T_{UU''}(fg)$ and $T_{UU}(1_U) = 1_{TU}$. The reader should unpack both diagrams into terms of hom–object morphisms and $\mathcal{V}$–functoriality. Composition of $\mathcal{V}$–2–functors is just composition of functions and components.

2.4. **Definition.** A *$\mathcal{V}$–2–natural transformation* $\alpha : T \to S : \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{W}}$ is a function sending each $U \in |\boldsymbol{\mathcal{U}}|$ to a $\mathcal{V}$–functor $\alpha_U : \mathcal{I} \to \boldsymbol{\mathcal{W}}(TU, SU)$ in such a way that we have commutativity of

$$\begin{array}{ccccc}
 & \mathcal{I} \otimes \boldsymbol{\mathcal{U}}(U,U') & \xrightarrow{\alpha_{U'} \otimes T_{UU'}} & \boldsymbol{\mathcal{W}}(TU',SU') \otimes \boldsymbol{\mathcal{W}}(TU,TU') & \\
{\scriptstyle =}\nearrow & & & & \searrow{\scriptstyle \mathcal{M}} \\
\boldsymbol{\mathcal{U}}(U,U') & & & & \boldsymbol{\mathcal{W}}(TU,SU') \\
{\scriptstyle =}\searrow & & & & \nearrow{\scriptstyle \mathcal{M}} \\
 & \boldsymbol{\mathcal{U}}(U,U') \otimes \mathcal{I} & \xrightarrow[S_{UU'} \otimes \alpha_U]{} & \boldsymbol{\mathcal{W}}(SU,SU') \otimes \boldsymbol{\mathcal{W}}(TU,SU) &
\end{array}$$

Unpacking this a bit, we see that α_U is an object $q = \alpha_U(0)$ in the $\mathcal{V}$–category $\boldsymbol{\mathcal{W}}(TU, SU)$ and a morphism $\alpha_{U_{00}} : I \to \boldsymbol{\mathcal{W}}(TU, SU)(q, q)$. By the $\mathcal{V}$–functoriality of α_U we see that $\alpha_{U_{00}} = j_q$. The axiom then states that $q'T_{UU'}(f) = S_{UU'}(f)q$ for all f, and that

$$\mathcal{M}_{(TU)(TU')(SU')_{(q',T_{UU'}(f))(q',T_{UU'}(g))}} \circ (j_{q'} \otimes T_{UU'_{fg}}) = \mathcal{M}_{(TU)(SU)(SU')_{(S_{UU'}(f),q)(S_{UU'}(g),q)}} \circ (S_{UU'_{fg}} \otimes j_q)$$

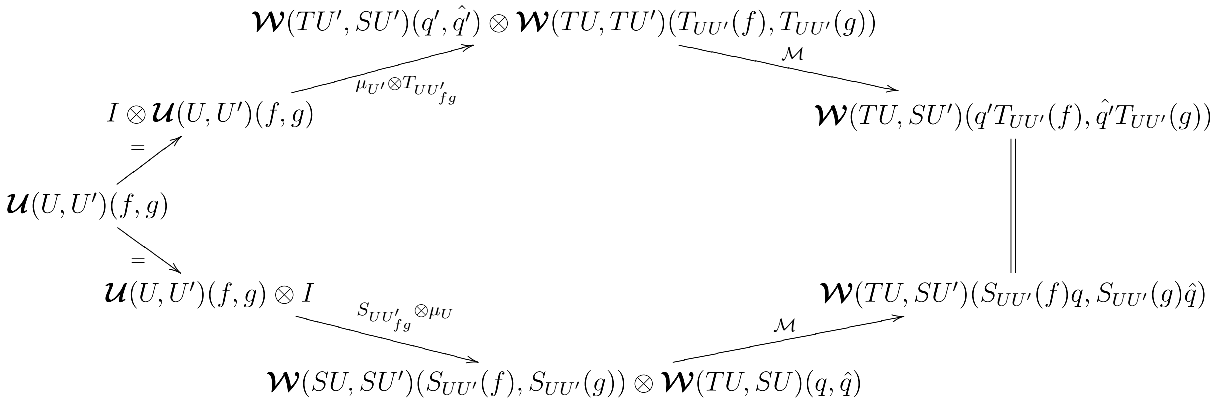
This is directly analogous to the usual definition of 2–natural transformation by whisker diagrams.

Vertical composition of $\mathcal{V}$–2–natural transformations is as expected. $(\beta \circ \alpha)_U =$

$$\begin{array}{c}
\mathcal{I} \otimes \mathcal{I} \\
\downarrow {\scriptstyle \beta_U \otimes \alpha_U} \\
\boldsymbol{\mathcal{W}}(SU,RU) \otimes \boldsymbol{\mathcal{W}}(TU,SU) \\
\downarrow {\scriptstyle \mathcal{M}} \\
\boldsymbol{\mathcal{W}}(TU,RU)
\end{array}$$

Identity 2–cells for vertical composition are $\mathcal{V}$–2–natural transformations $\mathbf{1}_T : T \to T$ where $(\mathbf{1}_T)_U = \mathcal{J}_{TU}$. Left and right whiskering of $\mathcal{V}$–2–functors onto $\mathcal{V}$–2–natural transformations are given by precisely the same descriptions as in the low dimensional case, with I replaced by $\mathcal{I}$, etc.

2.5. **Definition.** Given two $\mathcal{V}$–2–natural transformations a *$\mathcal{V}$–modification* between them $\mu : \theta \to \phi : T \to S : \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{W}}$ is a function that sends each object $U \in |\boldsymbol{\mathcal{U}}|$ to a morphism $\mu_U : I \to \boldsymbol{\mathcal{W}}(TU, SU)(\theta_U(0), \phi_U(0))$ in such a way that the following diagram commutes. (Let $\theta_U(0) = q$, $\phi_U(0) = \hat{q}$, $\theta_{U'}(0) = q'$ and $\phi_{U'}(0) = \hat{q}'$.)



This is directly analogous to the usual definition of modification described in section 1. Notice that since $\theta_{U_{00}} = j_{\theta_U(0)}$ for all $\mathcal{V}$–2–natural transformations θ we have that the morphism μ_U seen as a "family" consisting of a single morphism (corresponding to $0 \in |\mathcal{I}|$) constitutes a $\mathcal{V}$–natural transformation from θ_U to ϕ_U. "Vertical" compositions of modifications are given by the compositions of these underlying $\mathcal{V}$–natural transformations as described in section 1. Thus identities $\mathbf{1}_\alpha$ for this composition are families of $\mathcal{V}$–natural equivalences. Since α_U is a $\mathcal{V}$–functor from $\mathcal{I}$ to $\mathcal{W}(TU, SU)$ this means specifically that $((\mathbf{1}_\alpha)_U)_0 = j_{\alpha_U(0)} = j_q$.

2.6**. Theorem.** *$\mathcal{V}$–2–categories, $\mathcal{V}$–2–functors, $\mathcal{V}$–2–natural transformations and $\mathcal{V}$–modifications form a 3–category called $\mathcal{V}$–2–Cat.*

For proofs see either [Forcey2, 2004] or my thesis at
`http://scholar.lib.vt.edu/theses/available/etd-04232004-160123/`.

For $\mathcal{V}$ k–fold monoidal we have demonstrated that $\mathcal{V}$–Cat is $(k-1)$–fold monoidal. By induction we have that this process continues, i.e. that $\mathcal{V}$–n–Cat $= \mathcal{V}$–$(n-1)$–Cat–Cat is $(k-n)$–fold monoidal for $k > n$. For example, let us expand our description of the next level: the fact that $\mathcal{V}$–2–Cat $= \mathcal{V}$–Cat–Cat is $(k-2)$–fold monoidal. Now we are considering enrichment over $\mathcal{V}$–Cat. All the constructions in the proof above are recursively repeated. The unit $\mathcal{V}$–2–category is denoted as $\boldsymbol{\mathcal{I}}$ where $|\boldsymbol{\mathcal{I}}| = \{\mathbf{0}\}$ and $\boldsymbol{\mathcal{I}}(\mathbf{0}, \mathbf{0}) = \mathcal{I}$. Products of $\mathcal{V}$–2–categories are given by $\mathcal{U} \otimes \mathcal{W}$ for $i = 1...k-2$. Objects are pairs of objects as usual, and that there are exactly $k-2$ products is seen when the definition of hom–objects is given. In $\mathcal{V}$–2–Cat,

$$[\mathcal{U} \otimes_i^{(2)} \mathcal{W}]((U, W), (U', W')) = \mathcal{U}(U, U') \otimes \mathcal{W}(W, W')$$

Thus we have that

$$\begin{aligned}
&[\mathcal{U} \otimes \mathcal{W}]((U, W), (U', W'))((f, f'), (g, g')) \\
&= [\mathcal{U}(U, U') \otimes \mathcal{W}(W, W')]((f, f'), (g, g')) \\
&= \mathcal{U}(U, U')(f, g) \otimes \mathcal{W}(W, W')(f', g')
\end{aligned}$$

The definitions of $\alpha^{(2)}$ is just as in the lower case. For instance, $\alpha^{(2)}$ will now be a 3–natural transformation, that is, a family of $\mathcal{V}$–2–functors

$$\alpha^{(2)}_{\mathcal{U}\mathcal{V}\mathcal{W}} : (\mathcal{U} \otimes \mathcal{V}) \otimes \mathcal{W} \to \mathcal{U} \otimes (\mathcal{V} \otimes \mathcal{W}).$$

To each of these is associated a family of $\mathcal{V}$–functors

$$\alpha^{(2)}_{\boldsymbol{\mathcal{U}}\boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{W}}_{(U,V,W)(U',V',W')}} = \alpha^{(1)}_{\boldsymbol{\mathcal{U}}_{(U,U')}\boldsymbol{\mathcal{V}}_{(V,V')}\boldsymbol{\mathcal{W}}_{(W,W')}}$$

to each of which is associated a family of hom–object morphisms

$$\alpha^{(2)}_{\boldsymbol{\mathcal{U}}\boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{W}}_{(U,V,W)(U',V',W')_{(f,g,h)(f',g',h')}}} = \alpha_{\boldsymbol{\mathcal{U}}_{(U,U')(f,f')}\boldsymbol{\mathcal{V}}_{(V,V')(g,g')}\boldsymbol{\mathcal{W}}_{(W,W')(h,h')}}.$$

Now for the definitions of $\mathcal{V}$–n–categories and of the morphisms of $\mathcal{V}$–n–Cat.

3. Category of $\mathcal{V}$–n–Categories

The definition of a category enriched over $\mathcal{V}$–$(n-1)$–Cat is simply stated by describing the process as enriching over $\mathcal{V}$–$(n-1)$–Cat. In detail this means that:

3.1. **Definition.** A (small, strict) $\mathcal{V}$–n–*category* $\boldsymbol{\mathcal{U}}$ consists of

(1) A set of objects $|\boldsymbol{\mathcal{U}}|$

(2) For each pair of objects $A, B \in |\boldsymbol{\mathcal{U}}|$ a $\mathcal{V}$–$(n-1)$–category $\boldsymbol{\mathcal{U}}(A, B)$.

(3) For each triple of objects $A, B, C \in |\boldsymbol{\mathcal{U}}|$ a $\mathcal{V}$–$(n-1)$–functor

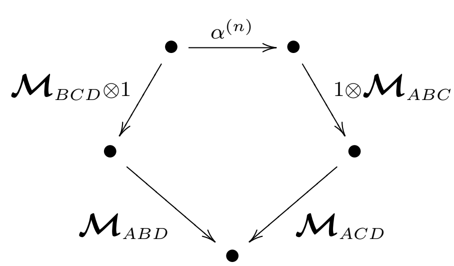
$$\boldsymbol{\mathcal{M}}_{ABC} : \boldsymbol{\mathcal{U}}(B, C) \otimes \boldsymbol{\mathcal{U}}(A, B) \to \boldsymbol{\mathcal{U}}(A, C)$$

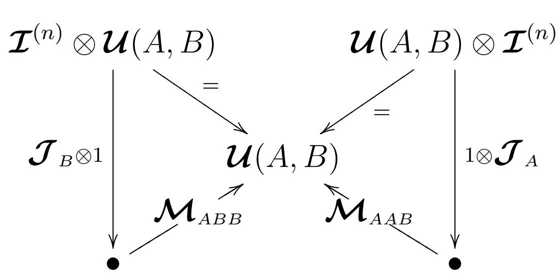
(4) For each object $A \in |\boldsymbol{\mathcal{U}}|$ a $\mathcal{V}$–$(n-1)$–functor

$$\boldsymbol{\mathcal{J}}_A : \boldsymbol{\mathcal{I}}^{(n-1)} \to \boldsymbol{\mathcal{U}}(A, A)$$

Henceforth we let the dimensions of domain for and particular instances of $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{J}}$ largely be determined by context.

(5) Axioms: The $\mathcal{V}$–$(n-1)$–functors that play the role of composition and identity obey commutativity of a pentagonal diagram (associativity axiom) and of two triangular diagrams (unit axioms). This amounts to saying that the functors given by the two legs of each diagram are equal.





The consequences of these axioms are expanded commuting diagrams just as in Example 2.1.

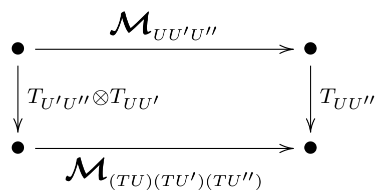
This definition requires that there be definitions of the unit $\boldsymbol{\mathcal{I}}^{(n)}$ and of $\mathcal{V}$–n–functors in place. First, from the proof of monoidal structure on $\mathcal{V}$–n–Cat, we can infer a recursively defined unit $\mathcal{V}$–n–category.

3.2**. Definition.** The unit object in $\mathcal{V}$–n–Cat is the $\mathcal{V}$–n–category $\boldsymbol{\mathcal{I}}^{(n)}$ with one object $\mathbf{0}$ and with $\boldsymbol{\mathcal{I}}^{(n)}(\mathbf{0},\mathbf{0}) = \boldsymbol{\mathcal{I}}^{(n-1)}$, where $\boldsymbol{\mathcal{I}}^{(n-1)}$ is the unit object in $\mathcal{V}$–$(n-1)$–Cat. Of course we let $\boldsymbol{\mathcal{I}}^{(0)}$ be I in $\mathcal{V}$. Also $\boldsymbol{\mathcal{M}}_{000} = \boldsymbol{\mathcal{J}}_0 = 1_{\boldsymbol{\mathcal{I}}^{(n)}}$.

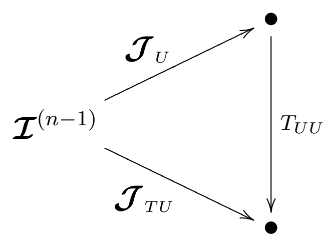
Now we can define the functors:

3.3**. Definition.** For two $\mathcal{V}$–n–categories $\boldsymbol{\mathcal{U}}$ and $\boldsymbol{\mathcal{W}}$ a *$\mathcal{V}$–n–functor* $T : \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{W}}$ is a function on objects $|\boldsymbol{\mathcal{U}}| \to |\boldsymbol{\mathcal{W}}|$ and a family of $\mathcal{V}$–$(n-1)$–functors $T_{UU'} : \boldsymbol{\mathcal{U}}(U,U') \to \boldsymbol{\mathcal{W}}(TU,TU')$. These latter obey commutativity of the usual diagrams.

(1) For $U, U', U'' \in |\boldsymbol{\mathcal{U}}|$



(2)



Here a $\mathcal{V}$–0–functor is just a morphism in $\mathcal{V}$.

$\mathcal{V}$–n–categories and $\mathcal{V}$–n–functors form a category. Composition of $\mathcal{V}$–n–functors is just composition of the object functions and composition of the hom–category $\mathcal{V}$–$(n-1)$–functors, with appropriate subscripts. Thus $(ST)_{UU'}(f) = S_{TUTU'}(T_{UU'}(f))$. Then it is straightforward to verify that the axioms are obeyed, as in

$$\begin{aligned}
&(ST)_{U'U''}(f)(ST)_{UU'}(g) \\
&= S_{TU'TU''}(T_{U'U''}(f))S_{TUTU'}(T_{UU'}(g)) \\
&= S_{TUTU''}(T_{U'U''}(f)T_{UU'}(g)) \\
&= S_{TUTU''}(T_{UU''}(fg)) \\
&= (ST)_{UU''}(fg).
\end{aligned}$$

That this composition is associative follows from the associativity of composition of the underlying functions and $\mathcal{V}$–$(n-1)$–functors. The 2–sided identity for this composition $1_{\boldsymbol{\mathcal{U}}}$ is made of the identity function (on objects) and identity $\mathcal{V}$–$(n-1)$–functor (for hom–categories.) The 1–cells we have just defined play a special role in the definition of a general k–cell for $k \geq 2$. These higher morphisms will be shown to exist and described in some detail in section 5.

4. Categorical Structure of $\mathcal{V}$–n–Cat

Now we demonstrate that $\mathcal{V}$–n–Cat has a special $(n+1)$–category structure that is a restriction of the morphisms occurring between images of the induced forgetful functors. This structure is the natural extension of the 2–category structure described for $\mathcal{V}$–Cat in [Kelly, 1982], and therefore when we speak of $\mathcal{V}$–n–Cat hereafter it will be the following structure to which we refer.

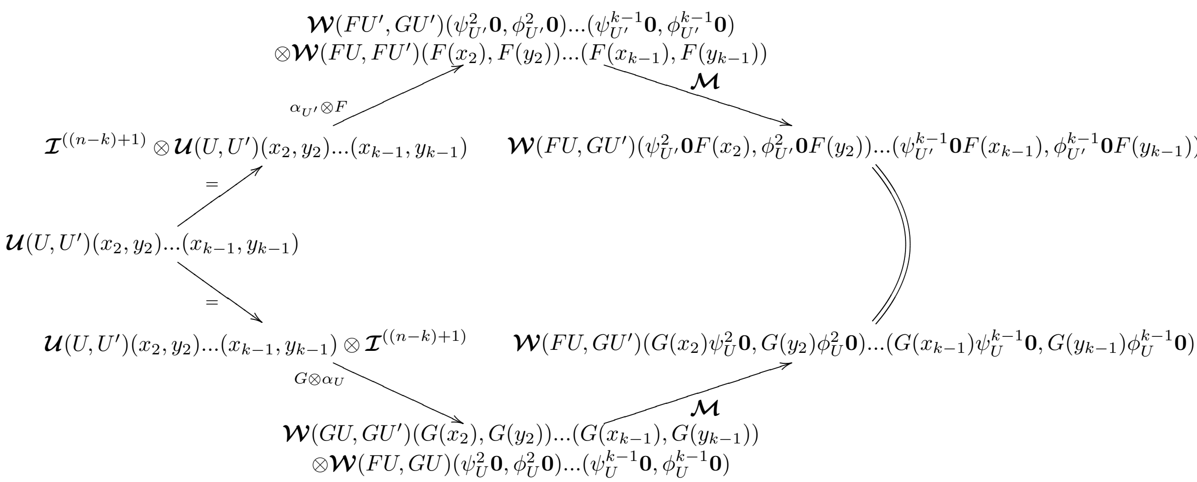
4.1**. Definition.** A $\mathcal{V}$*–n:k–cell* α between $(k-1)$–cells ψ^{k-1} and ϕ^{k-1}, written

$$\alpha : \psi^{k-1} \to \phi^{k-1} : \psi^{k-2} \to \phi^{k-2} : ... : \psi^2 \to \phi^2 : F \to G : \boldsymbol{\mathcal{U}} \to \boldsymbol{\mathcal{W}}$$

where F and G are $\mathcal{V}$–n–functors and where the superscripts denote cell dimension, is a function sending each $U \in |\boldsymbol{\mathcal{U}}|$ to a $\mathcal{V}$–$((n-k)+1)$–functor

$$\alpha_U : \boldsymbol{\mathcal{I}}^{((n-k)+1)} \to \boldsymbol{\mathcal{W}}(FU, GU)(\psi_U^2\mathbf{0}, \phi_U^2\mathbf{0})...(\psi_U^{k-1}\mathbf{0}, \phi_U^{k-1}\mathbf{0})$$

in such a way that we have commutativity of the following diagram. Note that the final (curved) equal sign is implied recursively by the diagram for the $(k-1)$–cells.



Thus for a given value of n there are k–cells up to $k = n+1$, making $\mathcal{V}$–n–Cat a potential $(n+1)$–category. We have already described composition of $\mathcal{V}$–n–functors. Now we describe all other compositions.

4.2**. Definition.** Case 1.

Let $k = 2 \ldots n+1$ and $i = 1 \ldots k-1$. Given α and β two $\mathcal{V}$–*n:k*–cells that share a common $\mathcal{V}$–*n:(k-i)*–cell γ, we can compose along the latter morphism as follows

$$(\beta \circ \alpha)_U = \boldsymbol{\mathcal{I}}^{((n-k)+1)} = \boldsymbol{\mathcal{I}}^{((n-k)+1)} \otimes \boldsymbol{\mathcal{I}}^{((n-k)+1)}$$

$$\downarrow \beta_U \otimes \alpha_U$$

$$\boldsymbol{\mathcal{W}}(FU,GU)(\psi_U^2\mathbf{0},\phi_U^2\mathbf{0})...(\psi_U^{k-i-1}\mathbf{0},\phi_U^{k-i-1}\mathbf{0})(\gamma_U\mathbf{0},\gamma''_U\mathbf{0})(\psi_U^{k-i+1}\mathbf{0},\phi_U^{k-i+1}\mathbf{0})...(\psi_U^{k-1}\mathbf{0},\phi_U^{k-1}\mathbf{0})$$

$$\otimes \boldsymbol{\mathcal{W}}_{(FU,GU)(\psi_U^2\mathbf{0},\phi_U^2\mathbf{0})...(\psi_U^{k-i-1}\mathbf{0},\phi_U^{k-i-1}\mathbf{0})(\gamma'_U\mathbf{0},\gamma_U\mathbf{0})(\delta_U^{k-i+1}\mathbf{0},\xi_U^{k-i+1}\mathbf{0})...(\delta_U^{k-1}\mathbf{0},\xi_U^{k-1}\mathbf{0})}$$

$$\downarrow \boldsymbol{\mathcal{M}}$$

$$\boldsymbol{\mathcal{W}}_{(FU,GU)(\psi_U^2\mathbf{0},\phi_U^2\mathbf{0})...(\psi_U^{k-i-1}\mathbf{0},\phi_U^{k-i-1}\mathbf{0})(\gamma'_U\mathbf{0},\gamma''_U\mathbf{0})(\psi_U^{k-i+1}\mathbf{0}\delta_U^{k-i+1}\mathbf{0},\phi_U^{k-i+1}\mathbf{0}\xi_U^{k-i+1}\mathbf{0})...(\psi_U^{k-1}\mathbf{0}\delta_U^{k-1}\mathbf{0},\phi_U^{k-1}\mathbf{0}\xi_U^{k-1}\mathbf{0})}$$

For α and β of different dimension and sharing a common cell of dimension lower than either the composition is accomplished by first raising the dimension of the lower of α and β to match the other by replacing it with a unit (see next Definition.)

Case 2.

It remains to describe composing along a 0–cell, i.e. along a common $\mathcal{V}$–n–category $\boldsymbol{\mathcal{W}}$. We describe composing a higher enriched cell with an enriched functor, and then leave the remaining possibilities to be accomplished by applying the first case to the results of such whiskering.

Composition with a $\mathcal{V}$–n–functor $K : \boldsymbol{\mathcal{W}} \to \boldsymbol{\mathcal{X}}$ on the right is given by:

$$(K\alpha)_U \;=\; \boldsymbol{\mathcal{I}}^{(n-k)+1}$$

$$\downarrow \alpha_U$$

$$\boldsymbol{\mathcal{W}}(FU,GU)(\psi_U^2\mathbf{0},\phi_U^2\mathbf{0})...(\psi_U^{k-1}\mathbf{0},\phi_U^{k-1}\mathbf{0})$$

$$\downarrow K_{FU,GU}$$

$$\boldsymbol{\mathcal{X}}(KFU,KGU)(K\psi_U^2\mathbf{0},K\phi_U^2\mathbf{0})...(K\psi_U^{k-1}\mathbf{0},K\phi_U^{k-1}\mathbf{0})$$

Composing with a $\mathcal{V}$–n–functor $H : \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{U}}$ on the left is given by $(\alpha H)_V = \alpha_{HV}$.

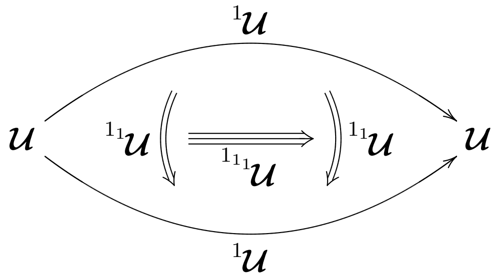
We describe unit k–cells for the above compositions.

4.3. **Definition.** A unit $\mathcal{V}$–*n:k–cell* $\mathbf{1}_{\psi^{k-1}}$ from a $(k-1)$–cell ψ^{k-1} to itself, sends each $U \in |\boldsymbol{\mathcal{U}}|$ to the $\mathcal{V}$–$((n-k)+1)$–functor

$$\boldsymbol{\mathcal{J}}_{\psi_U^{k-1}\mathbf{0}} : \boldsymbol{\mathcal{I}}^{((n-k)+1)} \to \boldsymbol{\mathcal{W}}(FU,GU)(\psi_U^2\mathbf{0},\phi_U^2\mathbf{0})...(\psi_U^{k-1}\mathbf{0},\psi_U^{k-1}\mathbf{0})$$

It is straightforward to verify that these fulfill the requirements of Definition 4.1 and indeed are units with respect to Definition 4.2.

Of course the unit for composition along a common cell of dimension more than 1 less than the composed cells is constructed of units for all the dimensions between that of the composed cells and that of the common cell. For example, the unit for composing along a common 0–cell may appear as follows:



4.4. **Theorem.** *$\mathcal{V}$–n–categories, $\mathcal{V}$–n–functors, and $\mathcal{V}$–n:k–cells for $k = 2 \ldots n+1$ together have the structure of an $(n+1)$–category.*

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