Consider the following subsets of $\mathbb{R}^3$

$$S = \left\{ \begin{bmatrix} 0 \\ x-y \\ 3y \end{bmatrix} \middle| \, x, y \in \mathbb{R} \right\}, \quad T = \left\{ \begin{bmatrix} x \\ 7y \\ y+3 \end{bmatrix} \middle| \, x, y \in \mathbb{R} \right\}, \quad U = \left\{ \begin{bmatrix} x \\ y \\ x^2+y^2 \end{bmatrix} \middle| \, x, y \in \mathbb{R} \right\}$$

Which is a subspace? Recall: subspaces are subsets that can be written as spans, and subspaces are planes or lines containing the origin $\mathbf{0}$ (or just the origin, or the whole space.)

$$S = \operatorname{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} \right\}$$

(T does not contain $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, U is not closed under addition)

2 Consider the following matrices: Find the Null space and Column space for each.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 6 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 7 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 10 \\ 0 & 0 & -3 & 0 & 6 \\ 0 & 0 & 4 & 4 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(augment each with $\vec{0}$)

A: $R_1 \leftrightarrow R_3$

$$\sim \left[\begin{array}{cccc|c} 3 & 0 & 0 & 0 & 0 \\ 6 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = x_2 \\ x_3 = 0 \\ x_4 = 0 \end{array}$$

$$N(A) = \operatorname{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \quad \operatorname{Col}(A) = \operatorname{Span}\left\{ \begin{pmatrix} 0 \\ 6 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

B: $R2 \leftarrow R2/2$, $R3 \leftarrow R3/-3$, $R4 \leftarrow R4/4$

$$\sim \left[\begin{array}{ccccc|c} 1 & 7 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & -35 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -39 & 0 \\ 0 & 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right]$$

$$N(B) = \operatorname{Span}\left\{ \begin{pmatrix} 39 \\ -5 \\ 2 \\ -2 \\ 1 \end{pmatrix} \right\} \quad \operatorname{Col}(B) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 4 \end{pmatrix} \right\}$$

C: $R3 \leftarrow R3 - 2R2$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = x_2 \\ x_3 = -\frac{1}{2} x_5 \\ x_4 = 0 \\ x_5 = x_5 \end{array}$$

$$N(C) = \operatorname{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\operatorname{Col}(C) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\}$$