* Chp. 5 Eigen-stuff

Def: When $T: V \to V$ is a lin. trans. and we find a specific vector $\vec{x} \in V$ such that $\vec{x} \neq \vec{0}$

and $T(\vec{x}) = c\vec{x}$ for some constant $c$

then we call $\vec{x}$ an eigenvector with eigen value $c$ (often use $c = \lambda$).
($\lambda$ can be 0, but $\vec{x} \neq \vec{0}$)

(if $T$ is just multiplying every vector by a constant, then every vector in $V$ is an eigenvector, with that constant $\lambda$ its eigenvalue.)

However, most lin. trans. $T: V \to V$ have only certain eigenvectors and eigenvalues. Find them!

Steps:

1) We work with $A = [T]_{\mathcal{B}}^{\mathcal{B}}$

2) Let $A\vec{x} = \lambda\vec{x}$ $(\vec{x} \neq \vec{0})$

then $\Rightarrow A\vec{x} = (\lambda I)\vec{x}$ $\left(\lambda I = \begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & & & \vdots \\ \vdots & & \ddots & & 0 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}\right)$

$\Rightarrow A\vec{x} - (\lambda I)\vec{x} = \vec{0}$

$\Rightarrow (A - \lambda I)\vec{x} = \vec{0}$

So $\vec{x} \neq \vec{0}$ and $\vec{x} \in N(A - \lambda I)$

$\Rightarrow \det(A - \lambda I) = 0$

3) this gives us an algebraic equation to solve for $\lambda$. Then plug back in to find $\vec{x}$.

ex) Let $T: \mathcal{P}^2 \to \mathcal{P}^2$
be given by $T(f(x)) = 2x f'(x) + 3x f''(x)$

Find the eigenvalues and their corresponding eigenvectors for $T$.

| $\vec{e}_i \in \mathcal{E}$ | $f'(x)$ | $f''(x)$ | $T(\vec{e}_i)$ |
|---|---|---|---|
| $1$ | $0$ | $0$ | $0$ |
| $x$ | $1$ | $0$ | $2x$ |
| $x^2$ | $2x$ | $2$ | $4x^2 + 6x$ |

1) $A = [T]_{\mathcal{E}}^{\mathcal{E}} = \left[ [0]_{\mathcal{E}} \; [2x]_{\mathcal{E}} \; [4x^2+6x]_{\mathcal{E}} \right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 4 \end{bmatrix}$

2) $\det(A - \lambda I) = \det\left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) = 0$

$= \det \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 4-\lambda \end{pmatrix} = 0$

$= -\lambda(2-\lambda)(4-\lambda) = 0$ ← This is called the characteristic (polynomial) equation.

$\Rightarrow \boxed{\lambda = 0, 2, 4}$

3) Solve $(A - \lambda I)\vec{x} = \vec{0}$

$\boxed{\lambda = 0}$

$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 2 & 6 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\begin{cases} x_1 = x_1 \text{ free} \\ x_2 = 0 \\ x_3 = 0 \end{cases}$

$\vec{x} \in \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} = \text{Span}\{1\}$

$\boxed{\lambda = 2}$

$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\begin{cases} x_1 = 0 \\ x_2 = x_2 \\ x_3 = 0 \end{cases}$

$\vec{x} \in \text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} = \text{Span}\{x\}$

$\boxed{\lambda = 4}$

$\left[\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 0 & -2 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{cases} x_1 = 0 \\ x_2 = 3x_3 \\ x_3 = x_3 \end{cases}$

$\vec{x} \in \text{Span}\left\{ \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\} = \text{Span}\{3x + x^2\}$

Note that the eigen vectors are found as spans. Indeed, for each eigen value $\lambda_0$ we get a subspace of dom$(T)$ called the $\boxed{\text{eigenspace}}$ $E_{\lambda_0}$. We find a basis for $E_{\lambda_0}$, so $E_{\lambda_0} = \text{span}\{\vec{x}_1, \vec{x}_2, \dots \vec{x}_k\}$.

→ We define the $\boxed{\text{geometric multiplicity}}$ of $\lambda_0$ as the dimension (number of basis vectors) $k$ of $E_{\lambda_0}$.

→ There is also the $\boxed{\text{algebraic multiplicity}}$ of $\lambda_0$, which is the power $p$ on the factor $(\lambda_0 - \lambda)^p$ in the characteristic polynomial $\det(A - \lambda I)$.

→ We can prove that for similar matrices $A$ and $B$, $B = P^{-1}AP$,

the eigenvalues are the same for both.

→ That's true for $[T]_{\mathcal{B}}^{\mathcal{B}}$ and $[T]_{\mathcal{C}}^{\mathcal{C}}$, two

matrices for the same lin. trans. $T: V \to V$ using two different bases, $\mathcal{B}$ and $\mathcal{C}$.

→ $T$ is $\boxed{\text{diagonalizable}}$ if there is a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}^{\mathcal{B}}$ is a diagonal matrix (any entry not on the main diagonal is zero).

→ Note that for a diagonal matrix, the eigenvalues are the diagonal entries.

Theorem: For $T: V \to V$

with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_j$,

the algebraic multiplicity of each $\lambda_i$ is equal to the corresponding geometric multiplicity of that $\lambda_i$, and the sum of those multiplicities totals to $n$,

<u>iff</u> $T$ is diagonalizable, that is, there is a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}^{\mathcal{B}}$ is diagonal.

Moreover, the diagonal entries of $[T]_{\mathcal{B}}^{\mathcal{B}}$ are the eigenvalues of $T$, with duplicates according to their algebraic multiplicities.

The basis $\mathcal{B}$ is the set of eigenvectors found by listing all the bases of the eigenspaces $E_{\lambda_i}$ together.

ex) $T(f(x)) = 2x f'(x) + 3x f''(x)$

$\lambda_1 = 0$, alg. mult. $= 1 =$ geom. mult. ✓

$\lambda_2 = 2$, alg. mult. $= 1 =$ geom. mult. ✓

$\lambda_3 = 4$, alg. mult $= 1 =$ geom. mult. ✓

diagonalizable!

$$\mathcal{B} = \{1, x, 3x + x^2\}, \quad [T]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Note: if $\lambda = 0$ is an eigenvalue of $T$ then $N(T) \neq \vec{0}$, and $T$ is not 1-1, not onto, and $\det([T]_{\mathcal{B}}^{\mathcal{B}}) = 0$.

ex) $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + y \\ 3y \end{pmatrix}$ diagonalizable?

$$A = [T]_{\mathcal{E}}^{\mathcal{E}} = \begin{bmatrix} 3(1)+0 & 3(0)+1 \\ 3(0) & 3(1) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix}\right) = 0$$

$$= (3-\lambda)(3-\lambda) = 0$$

$$= (3-\lambda)^2 = 0$$

$$\boxed{\lambda = 3}$$

power $p = 2$

Find eigenspace for $\lambda = 3$; Solve $(A - \lambda I)\vec{x} = \vec{0}$.

$$\left[\begin{array}{cc|c} 3-3 & 1 & 0 \\ 0 & 3-3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 = x_1 \text{ free} \\ x_2 = 0 \end{cases} \Rightarrow \vec{x} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

That is $E_3 = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

one vector in basis

So alg. mult. of $\lambda = 3$ is 2
geom mult. of $\lambda = 3$ is 1

$\Rightarrow$ Not diagonalizable.