Chp. 4 Linear Transformations

A $\boxed{\text{linear transformation}}$ is a function $T: V \to W$ that takes inputs from one vector space V and outputs vectors from another space W, and obeys: $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$
and: $T(c\vec{x}) = cT(\vec{x})$.

ex: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$
given by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x \\ -y \end{pmatrix}$

1) find $T\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ -2 \end{pmatrix}$

2) Show T is a lin. trans.

$$T\left(c\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} z \\ w \end{pmatrix}\right) = T\begin{pmatrix} cx+z \\ cy+w \end{pmatrix}$$

$$= \begin{pmatrix} cx+z+cy+w \\ 2(cx+z) \\ -(cy+w) \end{pmatrix}$$

$$= c\begin{pmatrix} x+y \\ 2x \\ -y \end{pmatrix} + \begin{pmatrix} z+w \\ 2z \\ -w \end{pmatrix}$$

$$= c\,T\begin{pmatrix} x \\ y \end{pmatrix} + T\begin{pmatrix} z \\ w \end{pmatrix}. \checkmark$$

T is a linear trans.

for $\vec{0} \in \mathbb{R}^2$, $T(\vec{0}) = T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0+0 \\ 2\,0 \\ -0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0} \in \mathbb{R}^3$

Note for $\vec{0} \in V$, $T(\vec{0}) = \vec{0} \in W$ always,
(if not, T is not linear trans.)

ex) (non example) $T: \mathbb{R}^2 \to \mathbb{R}^3$

given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+1 \\ 3y \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \vec{0} \quad \text{so not linear}$$

Also note $T\left(c\begin{pmatrix} x \\ y \end{pmatrix}\right) \neq c\,T\begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{pmatrix} cx+1 \\ 3cy \\ 0 \end{pmatrix} \neq \begin{pmatrix} cx+c \\ 3cy \\ 0 \end{pmatrix}$$

So, adding on constants is not linear.
Also squaring, sin and cos, e^x, ln are all non linear.

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ex) $T: P^3 \to P^3$

given by $T(f(x)) = f'(x) + 4f(x)$

find $T(2x^3 + 5x + 1)$

$$= 6x^2 + 5 + 8x^3 + 20x + 4$$
$$= 8x^3 + 6x^2 + 20x + 9$$

check that T is linear:

$$T(cf(x) + g(x)) = cf'(x) + g'(x) + 4(cf(x) + g(x))$$
$$= c(f'(x) + 4f(x)) + g'(x) + 4g(x)$$
$$= c\,T(f) + T(g). \checkmark$$

Every linear transformation can be represented by a matrix.

Given bases $\mathcal{B}$ for V, $\mathcal{C}$ for W

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots \vec{b}_n\},\ \mathcal{C} = \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_m\}$$

$T: V \to W$ is represented by a matrix $A_{m \times n} = [T]_{\mathcal{B}}^{\mathcal{C}}$

$$\boxed{[T]_{\mathcal{B}}^{\mathcal{C}} = A = \left[[T\vec{b}_1]_{\mathcal{C}}\ [T\vec{b}_2]_{\mathcal{C}}, \ldots, [T\vec{b}_n]_{\mathcal{C}} \right]}$$

so for $\vec{x} \in V$, we can find $T(\vec{x})$ by

1) finding $[\vec{x}]_{\mathcal{B}}$,
2) finding $A[\vec{x}]_{\mathcal{B}} = [T(\vec{x})]_{\mathcal{C}}$ (matrix times vector)
3) finding $T(\vec{x})$

ex: Find $[T]_{\mathcal{E}}^{\mathcal{E}}$ where $T: \mathcal{P}^3 \to \mathcal{P}^3$ is given by $T(f(x)) = f'(x) + 4f(x)$

$\mathcal{E} = \mathcal{E}_3 = \{1, x, x^2, x^3\}$,

$$A = [T]_{\mathcal{E}}^{\mathcal{E}} = \left[[0+4]_{\mathcal{E}}\ [1+4x]_{\mathcal{E}}\ [2x+4x^2]_{\mathcal{E}}\ [3x^2+4x^3]_{\mathcal{E}} \right]$$

$$= \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$T(2x^3+5x+1) = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{pmatrix} 1 \\ 5 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 20 \\ 6 \\ 8 \end{pmatrix} = 9 + 20x + 6x^2 + 8x^3$$

ex) $T: P^4 \to P^2$

given by $T(f(x)) = f''(x)$

$[T]_{\mathcal{E}}^{\mathcal{E}}$ uses $\mathcal{E}_4$ for inputs: $\{1, x, x^2, x^3, x^4\}$

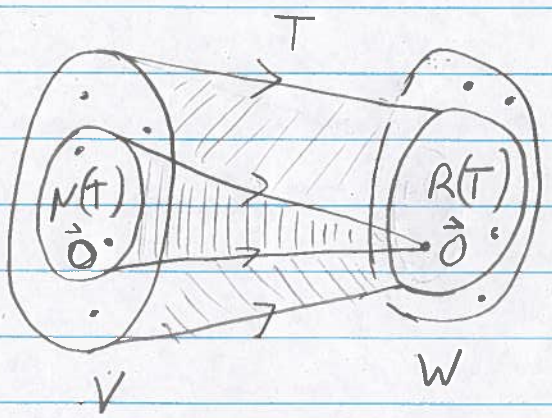
and $\mathcal{E}_2$ for outputs.

$$[T]_{\mathcal{E}}^{\mathcal{E}} = \Big[[0]_{\mathcal{E}} \;\; [0]_{\mathcal{E}} \;\; [2]_{\mathcal{E}} \;\; [6x]_{\mathcal{E}} \;\; [12x^2]_{\mathcal{E}} \Big]$$

$$= \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{bmatrix}, \quad 3 \times 5$$

Terminology: $T: V \to W$

- V is the domain, $\boxed{\text{dom}(T)}$
- W is the codomain, $\boxed{\text{codom}(T)}$
- Range (T) is a subspace of W, $\boxed{R(T)}$ which is all the outputs of T.
- Null Space of T, $\boxed{N(T)}$ is a subspace of V which is all the inputs that get taken to $\vec{0}$ by T.



- Null space is also known as kernel (T).

- Composition: for $T: V \to W$ and $S: W \to Y$ we make $S \circ T: V \to Y$ by $(S \circ T)(\vec{x}) = S(T(\vec{x}))$

$$V \xrightarrow[A]{T} W \xrightarrow[B]{S} Y$$

- If A represents T and B represents S (for same basis on W) then $S \circ T$ is represented by BA (matrix multiplication)

More terminology

- $T: V \to W$ is one-to-one (1-1) when each output has ~~only~~ exactly one input. For $\vec{y} \in R(T)$ if $T(\vec{a}) = \vec{y} = T(\vec{b})$ then $\vec{a} = \vec{b}$. (T is injective)

Theorem. T is one-to-one if and only if $N(T) = \{\vec{0}\}$.

Proof: Assume $N(T) = \{\vec{0}\}$.

Then if $T(\vec{a}) = T(\vec{b})$

$\Rightarrow T(\vec{a}) - T(\vec{b}) = \vec{0}$

$\Rightarrow T(\vec{a} - \vec{b}) = \vec{0}$ (linearity)

$\Rightarrow \vec{a} - \vec{b} = \vec{0}$ (by assumption)

$\Rightarrow \vec{a} = \vec{b}$

Next, Assume $N(T) \neq \{\vec{0}\}$, so $N(T) = \{\vec{0}, \vec{x}, \ldots\}$

then $T(\vec{0}) = \vec{0} = T(\vec{x})$, not 1-1. $\square$

- $T: V \to W$ is $\boxed{\text{onto}}$ (surjective) when $R(T) = W$.
- If T is 1-1 and onto, T is an $\boxed{\text{isomorphism}}$

Finding $N(T)$ and $R(T)$:

→ Same exact process as finding solution to $A\vec{x} = \vec{0}$ and $\text{col}(A)$, where $A = [T]_{\mathcal{B}}^{\mathcal{C}}$.

→ Find both: note that augment is $\vec{0}$

1) r.r. A to r.r.e.f.
   Recall: free variables are all non-pivot columns
2) write solution as a span, that's $N(T)$.
3) write $\text{col}(A)$ as a span of the original columns of A which correspond to pivots in r.r.e.f. That's $R(T)$.
4) Use bases $\mathcal{B}$ & $\mathcal{C}$ to describe $N(T)$ (using $\mathcal{B}$, the input basis) and $R(T)$ (using $\mathcal{C}$, the output basis.)

→ Note: since pivots + non-pivots = all the columns of A, we see that:

$$\dim(R(T)) + \dim(N(T)) = \dim(\text{dom}(T))$$

New terms: $T: V \to W$, $\dim V = n$, $\dim W = m$

→ $\boxed{\text{rank}(T)} = \text{rank}(A) = \dim(R(T))$

→ $\boxed{\text{nullity}(T)} = \text{nullity}(A) = \dim(N(T))$

→ So $\text{rank}(T) + \text{nullity}(T) = \dim(\text{dom}(T)) = n$ where n is also the number of columns of A

→ rank (A) = number of pivot columns of A

= number of (lin. indep.) vectors in any basis of col $(A) = R(T)$

→ nullity (A) = number of free variables in $A\vec{x} = \vec{0}$ solution

= number of (lin. indep) vectors in any basis of $N(T)$.

→ Note: if $N(T) = \{\vec{0}\}$ it has only one vector in it. The dimension is nullity $(T) = 0$, since $\{\vec{0}\}$ is not lin. indep.

$$\boxed{\begin{aligned} \to N(T) = \{\vec{0}\} &\Leftrightarrow \text{nullity}(T) = 0 \\ &\Leftrightarrow T \text{ is } 1\text{-}1 \\ &\Leftrightarrow \text{rank}(T) = \dim(\text{dom}(T)) = n \\ &\Leftrightarrow \text{columns of } A \text{ are lin. indep.} \end{aligned}}$$

$$\boxed{\begin{aligned} \to R(T) = \text{codom}(T) &\Leftrightarrow T \text{ is onto} \\ &\Leftrightarrow \text{rank}(T) = \dim(\text{codom}(T)) = m \\ &\Leftrightarrow \text{rows of } A \text{ are lin. indep.} \end{aligned}}$$

Also, if $T: V \to V$ is 1-1 and onto (bijective) so, an isomorphism, then T is invertible and $[T^{-1}]_{B}^{B} = A^{-1}$

where $A = [T]_{B}^{B}$.

So for square matrix A, $n \times n$:

$$
\begin{aligned}
\det(A) \neq 0 &\iff A \text{ is invertible} \\
&\iff T \text{ is 1-1} \quad (A = [T]_{B}^{B}) \\
&\iff T \text{ is onto} \\
&\iff \text{nullity}(T) = 0 \\
&\iff \text{rank}(T) = n \\
&\iff \text{rows of } A \text{ lin. indep.} \\
&\iff \text{columns of } A \text{ lin. indep.} \\
&\iff R(T) = W \\
&\iff N(T) = \{0\} \\
&\iff A\vec{x} = \vec{0} \text{ has one solution } \vec{0} \\
&\iff T \text{ is an isomorphism}
\end{aligned}
$$

ex) $T: \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(\vec{x}) = 5\vec{x}$.

$$
\begin{aligned}
A = [T]_{\mathcal{E}}^{\mathcal{E}} &= \left[\left[5\begin{pmatrix}1\\0\\0\end{pmatrix}\right]_{\mathcal{E}} \left[5\begin{pmatrix}0\\1\\0\end{pmatrix}\right]_{\mathcal{E}} \left[5\begin{pmatrix}0\\0\\1\end{pmatrix}\right]_{\mathcal{E}} \right] \\
&= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \\
&= 5I
\end{aligned}
$$

and $\det(A) = 125$
$\Rightarrow$ isomorphism

Recall our first example $T: \mathbb{R}^2 \to \mathbb{R}^3$

given by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x \\ -y \end{pmatrix}$

Find $[T]_{\mathcal{E}}^{\mathcal{E}} = A$, find rank, nullity, $N(T)$, $R(T)$.

$$[T]_{\mathcal{E}}^{\mathcal{E}} = \left[\left[T\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_{\mathcal{E}} \quad \left[T\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]_{\mathcal{E}} \right]$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -2 \end{bmatrix} = A_{3\times 2}, \quad m=3, \; n=2$$

Notice: this is just the matrix of coeffs of the system $\left\{\begin{matrix} x+y & = & \_ \\ 2x & = & \_ \\ -y & = & \_ \end{matrix}\right\}$ no constant yet.

A linear transformation just gives all the outputs of a system of linear functions.

r.r.e.f.
$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\uparrow \; \uparrow$ both pivots

so rank$(T) = 2 = n < m = 3$

nullity$(T) = 0$

$N(T) = \{\vec{0}\} = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$

$R(T) = \text{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}\right\}$

- one solution to $A\vec{x} = \vec{0}$
- T is 1-1
- T is not onto
- $R(T) \neq \mathbb{R}^3$
- rows are lin. dep.
- columns are lin. indep.

Two square matrices A, B both $n \times n$ are $\boxed{\text{similar}}$ when there exists a third matrix P which is square & invertible and $B = P^{-1} A P$

Ex: for a lin. trans. $T: V \to V$ and two bases $\mathcal{B}, \mathcal{C}$ of V

$$[T]_{\mathcal{B}}^{\mathcal{B}} = [I]_{\mathcal{C}}^{\mathcal{B}} [T]_{\mathcal{C}}^{\mathcal{C}} [I]_{\mathcal{B}}^{\mathcal{C}}$$

matrix rep. using basis $\mathcal{B}$ for input and output.

takes answer in $\mathcal{C}$ and switches to $\mathcal{B}$.

matrix rep. using $\mathcal{C}$

c. of b., or transition takes input in $\mathcal{B}$ and switches to $\mathcal{C}$

here $P = [I]_{\mathcal{B}}^{\mathcal{C}}$, $P^{-1} = [I]_{\mathcal{C}}^{\mathcal{B}}$

and similarity means "really the same transformation."

So, similar matrices have all the same: rank, nullity, 1-1, onto, eigenvalues.*

Also same determinants: $\det(P^{-1}AP) = \det(A)$.