Chp 3

Vector Spaces & Linear Transformations

$\mathbb{R}^m$, the vectors with m components, is an example of an m-dimensional vector space.

In general: a vector space over the real scalars is any set V with structures of addition and scaling: obeying: For $\vec{x}, \vec{y}, \vec{z} \in V$ and $c, d \in \mathbb{R}$

0) $\vec{x} + \vec{y} \in V$ and $c\vec{x} \in V$ closure
1) $(\vec{x} + \vec{y}) + z = \vec{x} + (\vec{y} + \vec{z})$ associative
2) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ commutative
3) there exists $\vec{0} \in V$ with $\vec{x} + \vec{0} = \vec{0} + \vec{x} = \vec{x}$ additive identity
4) there exists $-\vec{x} \in V$ with $\vec{x} + -\vec{x} = \vec{0}$ additive inverses
5) $c(d\vec{x}) = (cd)\vec{x}$ compatibility
6) $1\vec{x} = \vec{x}$ scalar identity
7) $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$ distributive
8) $(c + d)\vec{x} = c\vec{x} + d\vec{x}$ distributive

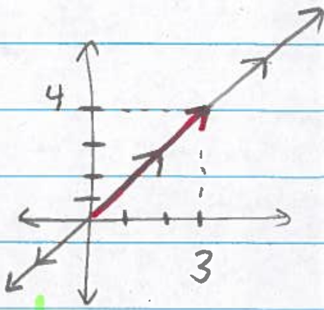
ex) R^m any m

ex) $M^{m \times n}$ all matrices m rows, n columns

ex) $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\}$ the set of all scalings of $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

that last one could be written:

$$S = \{ \vec{x} \in \mathbb{R}^2 \mid \vec{x} = c\begin{pmatrix}3\\4\end{pmatrix}, c \in \mathbb{R} \}$$



this S is a subspace of $\mathbb{R}^2$

Any subset of a vector space V which is closed under addition and scaling automatically will obey 1-8, so is a subspace.

$*$ for instance, any subspace contains $\vec{0}$

ex) $W = \{ \vec{x} \in \mathbb{R}^4 \mid \vec{x} = c_1\begin{pmatrix}2\\1\\0\\1\end{pmatrix} + c_2\begin{pmatrix}3\\4\\1\\0\end{pmatrix} \}$

check: W is closed, so it is a subspace.

Also, we define the $\boxed{\text{span}}$ of a set of vectors to be the set of all lin. combs of those vectors,

so $W = \text{Span}\left\{ \begin{pmatrix}2\\1\\0\\1\end{pmatrix}, \begin{pmatrix}3\\4\\1\\0\end{pmatrix} \right\}$

and $S = \text{Span}\left\{ \begin{pmatrix}3\\4\end{pmatrix} \right\}$

In fact, any subspace of a (finite dimensional) vector space can be written as the span of some of its vectors.

ex) For any matrix $A_{m\times n}$ the solution to $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{R}^n$.

- the solution contains $\vec{0}$.

We call this solution the Null Space $N(A)$.

ex) Find the null space $N(A)$ for

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

same as solve $A\vec{x} = \vec{0}$ ← augment

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \left.\begin{array}{l} x_1 = 0 \\ x_2 = x_2 \\ x_3 = 0 \end{array}\right\} \quad \vec{x} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow N(A) = \boxed{\text{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}}$$

e) Find the null space $N(B)$ for $B = \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 0 & 3 & 6 & 0 & -3 \end{bmatrix}$

$$\boxed{N(B) = \text{Span}\left\{\begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}}$$

solve $\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 3 & 6 & 0 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & -3 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 \end{array}\right]$

$$\Rightarrow \left.\begin{array}{l} x_1 - 3x_3 + 3x_5 = 0 \\ x_2 + 2x_3 - x_5 = 0 \\ x_3 = x_3 \\ x_4 = x_4 \\ x_5 = x_5 \end{array}\right\} \left.\begin{array}{l} x_1 = 3x_3 - 3x_5 \\ x_2 = -2x_3 + x_5 \\ x_3 = x_3 \\ x_4 = x_4 \\ x_5 = x_5 \end{array}\right\} \vec{x} = x_3 \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

ex) $S = \text{Span}\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$

is a subspace of $\mathbb{R}^2$.

But, that set of vectors is lin. dep.
(since $4 > 2$)
That means, some of those vectors can be made as lin. combs. of others, so the list is redundant: there is a smaller list whose span is S.

Def: a $\boxed{\text{basis}}$ $\mathcal{B}$ of a vector space V (or subspace) is a lin. indep. set of vectors $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ such that $\text{span}(\mathcal{B}) = V$.

To find a basis for S, row reduce the matrix of those column vectors.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ in r.r.e.f.}$$

→ find the pivots, and then find the original columns in those positions (col 1 and 3)

→ Then $S = \boxed{\text{Span}\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix} \right\}}$ for $\mathcal{B} = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix} \right\}$

→ ($\mathcal{B}$ is a basis)

→ This S is also called the $\boxed{\text{column space}}$ $\text{col}(A)$ of $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

ex) Find the column space, as span of a basis, for $A = \begin{bmatrix} 3 & 0 & 6 & 0 & 1 & 2 \\ 4 & 0 & 8 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 7 & 14 \end{bmatrix}$

r.r. $\sim \begin{bmatrix} 1 & 0 & 2 & 0 & 1/3 & 2/3 \\ 4 & 0 & 8 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 7 & 14 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 2 & 0 & 1/3 & 2/3 \\ 0 & 0 & 0 & 0 & -1/3 & -2/3 \\ 0 & 0 & 0 & 0 & 7 & 14 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 2 & 0 & 1/3 & 2/3 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 7 & 14 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
$\uparrow \qquad\qquad \uparrow$

$$\text{col}(A) = \text{Span}\left\{ \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix} \right\}$$

Note: if you know Calc 3; $\vec{n} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$

col(A) is a plane! $\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ 1 & 1 & 7 \end{vmatrix} = \langle 28, -21, -1 \rangle$

and $(0,0,0)$ is in the plane

so $28x - 21y - z = 0$ is an equation of the plane.

Also $\text{col}(A) = \text{Span}\left\{ \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix} \right\}$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{cases} x = 3t + s \\ y = 4t + s \\ z = 7s \end{cases} \right\}$$

is a parameterization of that plane.

Chp 3 cont

Def. The dimension of a vector space, $\dim(V)$ of V (or subspace S) is the number of vectors in any basis of V (or S).

ex) $\mathbb{R}^n$ has dimension n.
The standard basis for $\mathbb{R}^n$ is called $\mathcal{E} = \mathcal{E}(\mathbb{R}^n) = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$
where $\vec{e}_i$ = all zero components except one "1" in the i^{th} component.

$$\mathcal{E} \text{ for } \mathbb{R}^4 = \left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}$$

* $\mathcal{E}$ is ordered!

Note: $\mathbb{R}^n$ has many other bases (∞).

2-out-of-3 rule: if $\dim(V) = n$

set of $\boxed{n}$ vectors in V

set of vectors that $\boxed{\text{spans}}$ V

set of $\boxed{\text{lin. indep.}}$ vectors in V

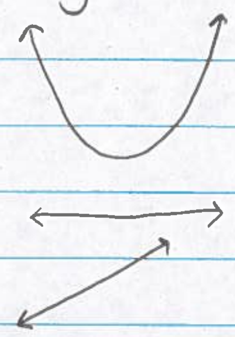
" any 2 of these implies the third!

- n lin. indep. vectors $\Rightarrow$ spans
- n vectors which span $\Rightarrow$ lin. indep.
- lin. indep. and spans $\Rightarrow$ exactly n vectors

A new vector space $\mathcal{P}^n$
is the set of all polynomials with degree at most n.

ex) $\mathcal{P}^2$ = all the polynomials with degree ≤ 2.

such as:
- x^2
- $3x^2 + 2$
- $\frac{1}{2}x^2 - 3x + 7$
- 5
- $x - 1$
- 0



always $= 0$, all x

$\mathcal{P}^2$ is a vector space: obeys all 8 axioms.

$\rightarrow \quad 3(3x^2+2) + (x-1) = 9x^2 + x + 5 \in \mathcal{P}^2$

$\rightarrow \quad 3x^2 + 2 + 0 = 3x^2 + 2$

ex) Is the set $\{x^2+3, x^2\}$ lin. indep?

Means: if $c_1(x^2+3) + c_2 x^2 = 0$
then is $c_1 = c_2 = 0$ the only solution?

Solve:

Expand $c_1 x^2 + c_1 3 + c_2 x^2 = 0$

$\Rightarrow (c_1 + c_2)x^2 + c_1 3 = 0$

But this must be true for all x-values, including $x = 0$!

$\Rightarrow \quad c_1 3 = 0$

$\Rightarrow \quad \boxed{c_1 = 0}$

$\Rightarrow \quad (0 + c_2)x^2 + 0(3) = 0$

$\Rightarrow \quad c_2 x^2 = 0$ true for $x = 1$!

$\Rightarrow \quad \boxed{c_2 = 0} \quad \Rightarrow$ lin. indep.

Also, the set $\mathcal{E} = \{1, x, x^2\} \subseteq \mathcal{P}^2$ is lin. indep. Plus, it spans $\mathcal{P}^2$! (any deg 2 or less polynomial looks like $f(x) = c + bx + ax^2$, written $[f(x)]_{\mathcal{E}} = \begin{pmatrix} c \\ b \\ a \end{pmatrix}$)

"column vector for $\mathcal{E}$"

So it is a basis for $\mathcal{P}^2$, and it has 3 items (vectors), so $\mathcal{P}^2$ has dimension = 3.

In general $\mathcal{P}^n$ has $\dim(\mathcal{P}^n) = n+1$ and standard basis $\mathcal{E}(\mathcal{P}^n)$
This $\mathcal{E}$ is also ordered:
$$\mathcal{E} = \{1, x, x^2, x^3, \dots, x^n\}.$$

An alternate basis for $\mathcal{P}^3$ is
$$\mathcal{B} = \{5,\ x^3+1,\ x^2,\ x+x^2\}$$
From now on, all our bases will have specific ordering.

Find $f(x) = 4x^3 + 2x$ as a lin. comb. of $\mathcal{B}$.

Method: in terms of standard basis $\mathcal{E}$ for $\mathcal{P}^3$
$$[f]_{\mathcal{E}} = f = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \end{pmatrix}, \quad \mathcal{B} = \left\{ \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

so solve (r.r.)
$$\left[\begin{array}{cccc|c} 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 5 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -4/5 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$f = -4/5\,(5) + 4(x^3+1) - 2(x^2) + 2(x+x^2). \quad \boxed{[f]_{\mathcal{B}} = \begin{pmatrix} -4/5 \\ 4 \\ -2 \\ 2 \end{pmatrix}}$$

Ex) Find $x^3 + 5x + 2 = f(x)$

in the basis $\mathcal{B} = \left\{ 1,\ x-3,\ \frac{(x-3)^2}{2},\ \frac{(x-3)^3}{6} \right\}$

That is, find $[f(x)]_{\mathcal{B}}$, the col. vector representation of f, in basis $\mathcal{B}$.

$$\mathcal{B} = \left\{ 1,\ x-3,\ \tfrac{1}{2}x^2 - 3x + \tfrac{9}{2},\ \tfrac{1}{6}x^3 - \frac{9x^2}{6} + \frac{27x}{6} - \frac{27}{6} \right\}$$

$$\left[\begin{array}{cccc|c} 1 & -3 & 9/2 & -9/2 & 2 \\ 0 & 1 & -3 & 9/2 & 5 \\ 0 & 0 & 1/2 & -3/2 & 0 \\ 0 & 0 & 0 & 1/6 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -9/2 & 9 & 17 \\ 0 & 1 & -3 & 9/2 & 5 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -9/2 & 17 \\ 0 & 1 & 0 & -9/2 & 5 \\ 0 & 0 & 1 & 0 & 18 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 44 \\ 0 & 1 & 0 & 0 & 32 \\ 0 & 0 & 1 & 0 & 18 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

so $\boxed{[f(x)]_{\mathcal{B}} = \begin{pmatrix} 44 \\ 32 \\ 18 \\ 6 \end{pmatrix}}$, $f(x) = 44 + 32(x-3) + 18\frac{(x-3)^2}{2} + 6\frac{(x-3)^3}{6}$

If you know calc II, that's the Taylor series for $f(x)$ at $x_0 = 3$.

$f(3) = 44$, $f'(3) = 3(3)^2 + 5 = 32$, $f''(3) = 18$, $f'''(3) = 6$

Ex) in $\mathbb{R}^2$, find $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ in the basis $\mathcal{B} = \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

$$\left[\begin{array}{cc|c} -2 & 1 & 3 \\ 1 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ -2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 7 \end{array}\right]$$

so $\boxed{\left[\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}}$; $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 2\begin{pmatrix} -2 \\ 1 \end{pmatrix} + 7\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ✓

Change of basis

For a given basis $\mathcal{B}$, the matrix to row reduce is always the same, only the augment changes.

Note: row reduction move on A gives the same result as:

(same r.r. move on I)$\cdot A$

ex: $A = \begin{bmatrix} 7 & 8 & 9 \\ 2 & 1 & 3 \\ 9 & 3 & 4 \end{bmatrix} \xrightarrow{R2 \leftarrow R2 + 2R3} \begin{bmatrix} 7 & 8 & 9 \\ 20 & 7 & 11 \\ 9 & 3 & 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 7 & 8 & 9 \\ 2 & 1 & 3 \\ 9 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 \\ 20 & 7 & 11 \\ 9 & 3 & 4 \end{bmatrix} \checkmark$$

So if we row reduce I with all the same moves, just like for finding A^{-1}, we'll get a matrix that can do these moves (via multiplication) on any vector. It will be a change-of-basis matrix from $\mathcal{E}$ to $\mathcal{B}$. We call it $[I]_{\mathcal{E}}^{\mathcal{B}}$.

ex)

$\mathcal{B} = \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

row reduce $\left[\begin{array}{cc|cc} -2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ -2 & 1 & 1 & 0 \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right]$ and $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ is the c.o.b.

$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$. $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} [\vec{x}]_{\mathcal{E}} = [\vec{x}]_{\mathcal{B}}$. $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = [I]_{\mathcal{E}}^{\mathcal{B}}$

For any two bases $\mathcal{B}$ and $\mathcal{C}$

we can find $[I]_{\mathcal{B}}^{\mathcal{C}}$. $\left(\begin{array}{l} \mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\} \\ \mathcal{C} = \{\vec{c}_1, \ldots, \vec{c}_n\} \end{array} \right)$

so $[I]_{\mathcal{B}}^{\mathcal{C}} [\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{C}}$

by $[I]_{\mathcal{B}}^{\mathcal{C}} = \Big[[\vec{b}_1]_{\mathcal{C}} \;\; [\vec{b}_2]_{\mathcal{C}} \;\; \ldots [\vec{b}_n]_{\mathcal{C}} \Big]$

columns are the basis vectors of $\mathcal{B}$, written as col. vectors in $\mathcal{C}$.

for our example $\rangle$ $[I]_{\mathcal{E}}^{\mathcal{B}} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \left[\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_{\mathcal{B}} \;\; \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]_{\mathcal{B}} \right]$

Note: $[I]_{\mathcal{B}}^{\mathcal{C}}$ is always square, $n \times n$.

$[I]_{\mathcal{B}}^{\mathcal{C}}$ is always invertible, and

$$\left([I]_{\mathcal{B}}^{\mathcal{C}}\right)^{-1} = [I]_{\mathcal{C}}^{\mathcal{B}}$$

example $\rangle$ $[I]_{\mathcal{B}}^{\mathcal{E}} = \left([I]_{\mathcal{E}}^{\mathcal{B}}\right)^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} = \left[\left[\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right]_{\mathcal{E}} \;\; \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_{\mathcal{E}} \right]$