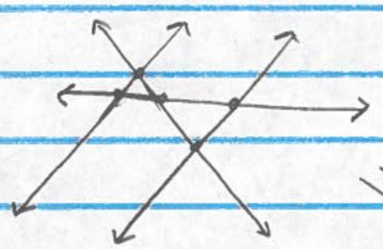
How to: Reverse engineer, to write a quiz!

Create a system that gives the picture-type:



augmented B: two pivots, random augment

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 5 \end{array}\right]$$

Notice: no set of 3 lines here has a solution!

do some random row reduction:

$R1 \leftarrow R1 + 2\cdot R2$

$R3 \leftarrow R3 - 3\cdot R2$

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & -3 & -2 \\ 0 & 0 & 5 \end{array}\right]$$

$R2 \leftarrow R2 + R1$

$R4 \leftarrow R4 - R1$

$R3 \leftarrow R3 + R1$

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 1 & 3 & 9 \\ 1 & -1 & 5 \\ -1 & -2 & -2 \end{array}\right] \Rightarrow \begin{cases} x + 2y = 7 \\ x + 3y = 9 \\ x - y = 5 \\ -x - 2y = -2 \end{cases}$$

System!

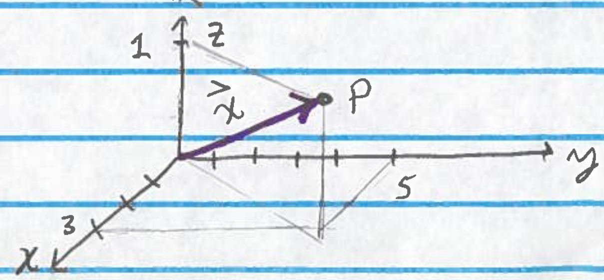
Now, solve that system the usual way, for practice.

Points & Vectors in $\mathbb{R}^n$

Any point in $\mathbb{R}^n$ can also be written as a (thought of as) a vector in $\mathbb{R}^n$.

$\mathbb{R}^3$ point $P = (3, 5, 1)$

vector $\vec{x} = \langle 3, 5, 1 \rangle = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$



Points are better for describing location, so the numbers are called <u>coordinates</u>. Vectors also describe location, but can also describe moving in that direction, or a force pulling in that direction, so the numbers are called <u>components</u>.

We can <u>add</u> <u>components</u> to add vectors, and <u>scale</u> vectors by multiplying components.

$$2\begin{pmatrix}3\\5\\1\end{pmatrix} = \boxed{\begin{pmatrix}6\\10\\2\end{pmatrix}}$$

$$\begin{pmatrix}3\\5\\1\end{pmatrix} + \begin{pmatrix}0\\-3\\6\end{pmatrix} = \boxed{\begin{pmatrix}3\\2\\7\end{pmatrix}}$$

Recall dot product $\langle 0,-3,6\rangle \cdot \langle 3,5,1\rangle = 0-15+6 = \boxed{-9}$

With variables: $\langle 3,5,1\rangle \cdot \langle x, y, z\rangle = \boxed{3x+5y+1z}$

Rows of coefficients $\begin{bmatrix}3 & 5 & 1\\ 2 & 0 & -4\end{bmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-9\\2\end{pmatrix}$

is a way to write the system

$$\begin{aligned} 3x + 5y + z &= -9 \\ 2x \qquad - 4z &= 2 \end{aligned}$$

so solve $\left[\begin{array}{ccc|c} 3 & 5 & 1 & -9 \\ 2 & 0 & -4 & 2 \end{array}\right]$ $R_1 \leftarrow R_1 - R_2$ $\left[\begin{array}{ccc|c} 1 & 5 & 5 & -11 \\ 2 & 0 & -4 & 2 \end{array}\right]$

$R_2 \leftarrow R_2 - 2R_1$ $\left[\begin{array}{ccc|c} 1 & 5 & 5 & -11 \\ 0 & -10 & -14 & -24 \end{array}\right]$ $\begin{array}{l} R_2 \leftarrow R_2/-10 \\ R_1 \leftarrow R_1 - 5R_2 \end{array}$ $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 1.4 & -2.4 \end{array}\right]$

$$\Rightarrow \quad \begin{aligned} x_1 - 2x_3 &= -1 \\ x_2 + 1.4x_3 &= -2.4 \\ x_3 &= x_3 \quad (\text{free!}) \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= -1 + 2x_3 \\ x_2 &= -2.4 - 1.4x_3 \\ x_3 &= x_3 \end{aligned}$$

OR
$$\boxed{\begin{aligned} x &= -1 + 2z \\ y &= -2.4 - 1.4z \\ z &= z \quad (\text{free!}) \end{aligned}}$$

general

specific
$$\boxed{\begin{aligned} x &= 1 \\ y &= -2.4 \\ z &= 0 \end{aligned}}$$

pick any value

OR
$$\boxed{\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} 2 \\ -1.4 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -2.4 \\ 0 \end{pmatrix}}$$

↳ this version of the general answer for a system is called a linear combination of constant vectors, with one variable coefficient. In general there is one vector for each free variable plus one constant vector (no variable).

Matrix operations : useful for short cuts.

1) Matrix times vector $A_{m \times n}$ times $\vec{x} \in \mathbb{R}^n$; $A\vec{x} \in \mathbb{R}^m$

→ mutiply components and sum (dot product) for each row of A (length n) and all of $\vec{x}$.

ex.

$A_{2\times 4}$, $\vec{x} \in \mathbb{R}^4$

$$\underset{A}{\begin{bmatrix} 1 & 3 & 0 & 2 \\ 5 & 4 & -1 & 0 \end{bmatrix}} \underset{\vec{x}}{\begin{pmatrix} 3 \\ 2 \\ 1 \\ -1 \end{pmatrix}} = \begin{pmatrix} 3+6+0-2 \\ 15+8-1+0 \end{pmatrix} = \begin{pmatrix} 7 \\ 22 \end{pmatrix}$$

$A\vec{x} \in \mathbb{R}^2$

2) Matrix times matrix

$A_{m\times n}$ times $B_{n\times q}$ gives AB, $m\times q$.

→ find the entries of AB (in say row i and column j) by multiplying and summing (dot product) row i of A times column j of B.

formula: $$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

ex: $$\underbrace{\begin{bmatrix} 3 & 0 & 1 & 2 \\ 4 & -1 & 0 & 1 \end{bmatrix}}_{A_{2\times 4}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix}}_{B_{4\times 3}}$$

$$\underset{(2\times 3)}{AB} = \begin{bmatrix} 3+0+1+4 & 0+0-1+0 & 6+0+0-2 \\ 4+0+0+2 & 0-1+0+0 & 8-1+0-1 \end{bmatrix} = \begin{bmatrix} 8 & -1 & 4 \\ 6 & -1 & 6 \end{bmatrix}$$

3) Matrix + Matrix, scalar times matrix

A, B both $m\times n$

$$\begin{bmatrix} 3 & 2 & 0 \\ 4 & 1 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 3 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 3 \\ 3 & 2 & -2 \end{bmatrix}$$

$(A+B)_{ij} = A_{ij} + B_{ij}$

$$5\begin{bmatrix} 3 & 2 & 0 \\ 4 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 10 & 0 \\ 20 & 5 & -10 \end{bmatrix}$$

$(cA)_{ij} = c(A_{ij})$

4) Matrix transpose: $A_{m\times n} \longrightarrow A^t_{n\times m}$

$$(A^t)_{ij} = A_{ji} \qquad \text{"rows become columns"}$$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 0 \end{bmatrix} \longrightarrow A^t = \begin{bmatrix} 3 & 4 \\ 2 & 5 \\ 1 & 0 \end{bmatrix}$$

5) Matrix determinant: $A_{n\times n} \longrightarrow \det(A) \in \mathbb{R}$

(square A) → scalar,

$$\det A = \sum_{j=1}^{n} (-1)^{j+1} A_{1j} \det(M_{1j})$$

where M_{1j} is the matrix made from A by deleting row 1 and column j.

→ Recursive! We also need: $\det(c) = c$ for any scalar c, which is a 1×1 matrix.

$$\det \begin{bmatrix} 3 & 1 & 2 \\ 4 & 5 & 6 \\ 0 & -3 & -1 \end{bmatrix} \begin{cases} n = 3 \\ k = 1 \text{ to } 3 \end{cases}$$

$$= (-1)^2 3 \det \begin{bmatrix} 5 & 6 \\ -3 & -1 \end{bmatrix} + (-1)^3 1 \det \begin{bmatrix} 4 & 6 \\ 0 & -1 \end{bmatrix} + (-1)^4 2 \det \begin{bmatrix} 4 & 5 \\ 0 & -3 \end{bmatrix}$$

$$= 3\left((-1)^2 5 \det(-1) + (-1)^3 6 \det(-3)\right)$$
$$+ -1\left((-1)^2 4 \det(-1) + (-1)^3 6 \det(0)\right)$$
$$+ 2\left((-1)^2 4 \det(-3) + (-1)^3 5 \det(0)\right)$$

$$= 3(5(-1) - 6(-3)) - (4(-1) - 6(0)) + 2(4(-3) - 5(0))$$

$$= 3(13) + 4 + 2(-12)$$

$$= \boxed{19}$$

- Using any row i

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(M_{ij})$$

going along row i

$(-1)^{i+j}$ has checkerboard pattern

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \\ + & - & + & - & + & \\ \vdots & & & & & \ddots \end{bmatrix}$$

(odd + odd = even)

- or you can use a column!

- So, if A has a row of zeros, or a column of zeros, then $\det(A) = 0$.

- If A is triangular (either all zeros above or below the main diagonal (upper left to lower right) then $\det(A)$ = multiplying all the main diagonal entries A_{ii}.

$$\det \begin{bmatrix} 3 & 0 & 4 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -4 \end{bmatrix} = -24$$

More determinant facts and shortcuts

- 2x2 $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

- row equivalence moves change the det.

1) Switching 2 rows $\rightarrow$ multiply by -1.

$A \sim B$ by $R_i \leftrightarrow R_k$

$\Rightarrow \det(A) = (-1)\det(B)$

2) Scaling 1 row $\rightarrow$ multiply by: $1/\text{scalar}$

$A \sim B$ by $R_i \leftarrow cR_i$

$\Rightarrow \det(A) = \frac{1}{c}\det(B)$

3) Adding a multiple of one row to another $\rightarrow$ no change

$A \sim B$ by $R_i \leftarrow R_i + cR_k$

$\Rightarrow \det(B) = \det(A)$

- $\det(A^t) = \det(A)$

- $\det(AB) = \det(A)\det(B)$

Note:
$AB \neq BA$
but
$\det(AB) = \det(BA)$

ex)
$$\det\begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \overset{R1 \leftrightarrow R3}{=} (-1)\det\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad R3 \leftarrow \tfrac{1}{2}R3$$

$$(-1)\,2\det\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \overset{R3 \leftarrow R3 - R1}{=} -2\det\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -3 & -2.5 \\ 0 & 0 & 0 & 2 \end{bmatrix} = -2(-6)$$

$$= \boxed{12}$$

Identity matrix and inverse matrix

→ The identity matrix $I_{n \times n}$, sometimes written I_n or just I, has entries
1 on the main diagonal
0 off the main diagonal

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

→ For a square matrix $A_{n \times n}$

$$AI = IA = A$$ row times column

→ Only some square matrices $A_{n \times n}$ are invertible; which means that there exists another square matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I$$

→ A is invertible if and only if $\det(A) \neq 0$.

→ to find A^{-1}, augment A with I (all at the same time) and row reduce

$$[A \mid I] \sim [I \mid A^{-1}]$$

ex) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$, $\det(A) = -1$

find A^{-1}: $\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right]$

$A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & -1 \end{bmatrix}$ check $AA^{-1} = I = A^{-1}A$

- $\det(A^{-1}) = 1/\det(A)$

- When A^{-1} does not exist, we say A is singular (or non-invertible)

- For invertible A, augmenting A with a column of constants (vector) $\vec{b}$ is the same as solving a system with variables $(x_1, x_2, \ldots, x_n) = \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and n equations.

  System: $A\vec{x} = \vec{b}$

Then the solution (one unique solution) is $\vec{x} = A^{-1}\vec{b}$.

Handout: solving $A\vec{x} = \vec{b}$.