More determinant facts and shortcuts

- 2x2 $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

- row equivalence moves change the det.

1) Switching 2 rows $\rightarrow$ multiply by -1.

$A \sim B$ by $R_i \leftrightarrow R_k$

$\Rightarrow \det(A) = (-1)\det(B)$

2) Scaling 1 row $\rightarrow$ multiply by: $1/\text{scalar}$

$A \sim B$ by $R_i \leftarrow cR_i$

$\Rightarrow \det(A) = \frac{1}{c}\det(B)$

3) Adding a multiple of one row to another $\rightarrow$ no change

$A \sim B$ by $R_i \leftarrow R_i + cR_k$

$\Rightarrow \det(B) = \det(A)$

- $\det(A^t) = \det(A)$

- $\det(AB) = \det(A)\det(B)$

Note:
$AB \neq BA$
but
$\det(AB) = \det(BA)$

ex) $\det\begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \overset{R1 \leftrightarrow R3}{=} (-1)\det\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ $R3 \leftarrow \frac{1}{2}R3$

$(-1)\,2\det\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \overset{R3 \leftarrow R3 - R1}{=} -2\det\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -3 & -2.5 \\ 0 & 0 & 0 & 2 \end{bmatrix} = -2(-6)$

$= \boxed{12}$

Identity matrix and inverse matrix

→ The identity matrix $I_{n \times n}$, sometimes written I_n or just I, has entries
1 on the main diagonal
0 off the main diagonal

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

→ For a square matrix $A_{n \times n}$

$$AI = IA = A$$ row times column

→ Only some square matrices $A_{n \times n}$ are invertible; which means that there exists another square matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I$$

→ A is invertible if and only if $\det(A) \neq 0$.

→ to find A^{-1}, augment A with I (all at the same time) and row reduce

$$[A \mid I] \sim [I \mid A^{-1}]$$

ex) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$, $\det(A) = -1$

find A^{-1}: $\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right]$

$A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & -1 \end{bmatrix}$ check $AA^{-1} = I = A^{-1}A$

- $\det(A^{-1}) = 1/\det(A)$

- When A^{-1} does not exist, we say A is singular (or non-invertible)

- For invertible A, augmenting A with a column of constants (vector) $\vec{b}$ is the same as solving a system with variables $(x_1, x_2, \dots, x_n) = \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and n equations.

  System: $A\vec{x} = \vec{b}$

  Then the solution (one unique solution) is $\vec{x} = A^{-1}\vec{b}$.

Handout: solving $A\vec{x} = \vec{b}$.