Back to solution method: matrix $A_{m \times n}$ has • m rows • n columns

→ Recall, from a system of (affine) linear equations we write a matrix (augmented) of scalar coefficients and solve using $\boxed{\text{row reduction moves.}}$

→ Two matrices are $\boxed{\text{row equivalent,}}$ $A \sim B$, when you get from A to B by row reduction moves.

→ a $\boxed{\text{pivot}}$ in a matrix B is a "1" in a row of B with • all "0"s before it, in its row
and • all "0"s above and below, in its column

→ The $\boxed{\text{row reduced echelon form}}$ of A (r.r.e.f.) is a matrix $B \sim A$ where each row of B is either all 0's or has a pivot 1
and the pivots in earlier (higher) rows are in earlier (further left) columns, and "0" rows are at bottom.

ex:

$$B = \left.\begin{bmatrix} 0 & 0 & 1 & 0 & 2 & 0 & 3 & 0 & 2 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 5 & -2 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right\} \text{lots of numbers after (but not above or below) pivots}$$

3 pivots, and one row all "0"

→ If $A \sim B$ in r.r.e.f, a $\boxed{\text{pivot column of } A}$ is a column of A where that column in B has a pivot

→ a system is solved when its matrix A of coefficients is put in r.r.e.f. B (the moves are also done on the augmented column of constants, but that column doesn't have to be in r.r.e.f.)

Then the r.r.e.f. B is returned to equations as follows:

- each column corresponds to an original variable x, y, z or $x_1, x_2, x_3, x_4, \dots$ (except the augment column, which is constants).
- each pivot in B is a determined variable of the solution: it will be on the left of an equation.
- each non-pivot column of B is a free variable, it can be any real number.

ex:

$$
\begin{array}{c}
\overbrace{\hphantom{0\;1\;0\;0\;-2\;0\;5}}^{B} \quad (\text{augment}) \\
\left[\begin{array}{ccccccc|c}
0 & 1 & 0 & 0 & -2 & 0 & 5 & 3 \\
0 & 0 & 0 & 1 & 1 & 0 & 3 & 1/4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \\
\begin{array}{ccccccc}
x_1 & \boxed{x_2} & x_3 & \boxed{x_4} & x_5 & x_6 & x_7
\end{array}
\end{array}
\longrightarrow
\begin{array}{ll}
x_1 = x_1 & (\text{free!}) \\
x_2 - 2x_5 + 5x_7 = 3 & \\
x_3 = x_3 & (\text{free!}) \\
x_4 + x_5 + 3x_7 = 1/4 & \\
x_5 = x_5 & (\text{free!}) \\
x_6 = x_6 & (\text{free!}) \\
x_7 = x_7 & (\text{free!})
\end{array}
$$

pivots

Next we solve the non-free equations, one for each pivot.

Five free variables = 5 dimensional solution

→ $$\boxed{\begin{aligned} x_1 &= x_1 \\ x_2 &= 3 + 2x_5 - 5x_7 \\ x_3 &= x_3 \\ x_4 &= 1/4 - x_5 - 3x_7 \\ x_5 &= x_5 \\ x_6 &= x_6 \\ x_7 &= x_7 \end{aligned}}$$

This is the final general solution. There are ∞ solution points since choosing any values for the free variables gives a specific solution.

Specific solution example:

$x_1 = 0$ ← pick any!

$x_2 = ?$ ← find: $3 + 2(-2) - 5(0) = -1$

$x_3 = 1$ ← pick any!

$x_4 = ?$ ← find: $\frac{1}{4} - (-2) - 3(0) = \frac{9}{4}$

$x_5 = -2$ ← pick any!

$x_6 = 3$ ← pick any!

$x_7 = 0$ ← pick any!

$$\boxed{\begin{aligned} x_1 &= 0 \\ x_2 &= -1 \\ x_3 &= 1 \\ x_4 &= 9/4 \\ x_5 &= -2 \\ x_6 &= 3 \\ x_7 &= 0 \end{aligned}}$$