New terms: $T: V \to W$, $\dim V = n$, $\dim W = m$

→ $\boxed{\text{rank}(T)} = \text{rank}(A) = \dim(R(T))$

→ $\boxed{\text{nullity}(T)} = \text{nullity}(A) = \dim(N(T))$

→ So $\text{rank}(T) + \text{nullity}(T) = \dim(\text{dom}(T)) = n$
where n is also the number of columns of A

→ rank (A) = number of pivot columns of A
= number of (lin. indep.) vectors in any basis of $\text{col}(A) = R(T)$

→ nullity (A) = number of free variables in $A\vec{x} = \vec{0}$ solution
= number of (lin. indep) vectors in any basis of $N(T)$.

→ Note: if $N(T) = \{\vec{0}\}$ it has only one vector in it. The dimension is nullity $(T) = 0$, since $\{\vec{0}\}$ is not lin. indep.

→ $N(T) = \{\vec{0}\} \Leftrightarrow \text{nullity}(T) = 0$
$\Leftrightarrow$ T is 1-1
$\Leftrightarrow \text{rank}(T) = \dim(\text{dom}(T)) = n$
$\Leftrightarrow$ columns of A are lin. indep.

→ $R(T) = \text{codom}(T) \Leftrightarrow T$ is onto
$\Leftrightarrow \text{rank}(T) = \dim(\text{codom}(T)) = m$
$\Leftrightarrow$ rows of A are lin. indep.

Also, if $T: V \to V$ is 1-1 and onto (bijective) so, an isomorphism, then T is invertible and $[T^{-1}]_{\mathcal{B}}^{\mathcal{B}} = A^{-1}$

where $A = [T]_{\mathcal{B}}^{\mathcal{B}}$.

So for square matrix A, $n \times n$:

$$
\begin{aligned}
\det(A) \neq 0 &\iff A \text{ is invertible} \\
&\iff T \text{ is } 1\text{-}1 \quad (A = [T]_{\mathcal{B}}^{\mathcal{B}}) \\
&\iff T \text{ is onto} \\
&\iff \text{nullity}(T) = 0 \\
&\iff \text{rank}(T) = n \\
&\iff \text{rows of } A \text{ lin. indep.} \\
&\iff \text{columns of } A \text{ lin. indep.} \\
&\iff R(T) = W \\
&\iff N(T) = \{0\} \\
&\iff A\vec{x} = \vec{0} \text{ has one solution } \vec{0} \\
&\iff T \text{ is an isomorphism}
\end{aligned}
$$

ex) $T: \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(\vec{x}) = 5\vec{x}$.

$$A = [T]_{\mathcal{E}}^{\mathcal{E}} = \left[\left[5\begin{pmatrix}1\\0\\0\end{pmatrix}\right]_{\mathcal{E}} \left[5\begin{pmatrix}0\\1\\0\end{pmatrix}\right]_{\mathcal{E}} \left[5\begin{pmatrix}0\\0\\1\end{pmatrix}\right]_{\mathcal{E}} \right]$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$ and $\det(A) = 125$ $\Rightarrow$ isomorphism

$$= 5I$$

Recall our first example $T: \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x \\ -y \end{pmatrix}$$

Find $[T]_{\mathcal{E}}^{\mathcal{E}} = A$, find rank, nullity, $N(T)$, $R(T)$.

$$[T]_{\mathcal{E}}^{\mathcal{E}} = \left[\left[T\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_{\mathcal{E}} \quad \left[T\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]_{\mathcal{E}} \right]$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & -2 \end{bmatrix} = A_{3\times 2} \qquad m=3,\ n=2$$

Notice: this is just the matrix of coeffs of the system $\left\{\begin{matrix} x+y = \_ \\ 2x \quad = \_ \\ -y = \_ \end{matrix}\right\}$ no constant yet.

A linear transformation just gives all the outputs of a system of linear functions.

r.r.e.f. $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & -2 & 0 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\uparrow \ \uparrow$ both pivots

so rank $(T) = 2 = n < m = 3$

nullity $(T) = 0$

$N(T) = \{\vec{0}\} = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$

$R(T) = \text{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}\right\}$

- one solution to $A\vec{x} = \vec{0}$
- T is 1-1
- T is not onto
- $R(T) \neq \mathbb{R}^3$
- rows are lin. dep.
- columns are lin. indep.

Two square matrices A, B both $n \times n$ are $\boxed{\text{similar}}$ when there exists a third matrix P which is square & invertible and $B = P^{-1}AP$

Ex: for a lin. trans. $T: V \to V$ and two bases $\mathcal{B}, \mathcal{C}$ of V

$$[T]_{\mathcal{B}}^{\mathcal{B}} = [I]_{\mathcal{C}}^{\mathcal{B}} [T]_{\mathcal{C}}^{\mathcal{C}} [I]_{\mathcal{B}}^{\mathcal{C}}$$

matrix rep. using basis $\mathcal{B}$ for input and output.

takes answer in $\mathcal{C}$ and switches to $\mathcal{B}$.

matrix rep. using $\mathcal{C}$

c. of b., or transition takes input in $\mathcal{B}$ and switches to $\mathcal{C}$

here $P = [I]_{\mathcal{B}}^{\mathcal{C}}$, $P^{-1} = [I]_{\mathcal{C}}^{\mathcal{B}}$

and similarity means "really the same transformation."

So, similar matrices have all the same:
rank, nullity, 1-1, onto, eigenvalues.*

Also same determinants: $\det(P^{-1}AP) = \det(A)$.