Chp. 4 Linear Transformations

A $\boxed{\text{linear transformation}}$ is a function $T: V \to W$ that takes inputs from one vector space V and outputs vectors from another space W, and obeys: $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$

and: $T(c\vec{x}) = cT(\vec{x})$.

ex: $T: \mathbb{R}^2 \to \mathbb{R}^3$

given by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x \\ -y \end{pmatrix}$

1) find $T\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ -2 \end{pmatrix}$

2) Show T is a lin. trans.

$$T\left(c\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} z \\ w \end{pmatrix}\right) = T\begin{pmatrix} cx+z \\ cy+w \end{pmatrix}$$

$$= \begin{pmatrix} cx+z+cy+w \\ 2(cx+z) \\ -(cy+w) \end{pmatrix}$$

$$= c\begin{pmatrix} x+y \\ 2x \\ -y \end{pmatrix} + \begin{pmatrix} z+w \\ 2z \\ -w \end{pmatrix}$$

$$= cT\begin{pmatrix} x \\ y \end{pmatrix} + T\begin{pmatrix} z \\ w \end{pmatrix}. \checkmark$$

T is a linear trans.

for $\vec{0} \in \mathbb{R}^2$,

$$T(\vec{0}) = T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0+0 \\ 2 \cdot 0 \\ -0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0} \in \mathbb{R}^3$$

Note for $\vec{0} \in V$, $T(\vec{0}) = \vec{0} \in W$ always.
(if not, T is not linear trans.)

ex) (non example) $T: \mathbb{R}^2 \to \mathbb{R}^3$

given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+1 \\ 3y \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \vec{0} \quad \text{so not linear}$$

Also note $T\left(c\begin{pmatrix} x \\ y \end{pmatrix}\right) \neq c\,T\begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{pmatrix} cx+1 \\ 3cy \\ 0 \end{pmatrix} \neq \begin{pmatrix} cx+c \\ 3cy \\ 0 \end{pmatrix}$$

So, adding on constants is not linear. Also squaring, sin and cos, e^x, ln are all non linear.

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ex) $T: P^3 \to P^3$

given by $T(f(x)) = f'(x) + 4f(x)$

Find $T(2x^3 + 5x + 1)$

$$= 6x^2 + 5 + 8x^3 + 20x + 4$$
$$= 8x^3 + 6x^2 + 20x + 9$$

check that T is linear:

$$T(cf(x) + g(x)) = cf'(x) + g'(x) + 4(cf(x) + g(x))$$
$$= c(f'(x) + 4f(x)) + g'(x) + 4g(x)$$
$$= c\,T(f) + T(g). \checkmark$$

Every linear transformation can be represented by a matrix.

Given bases $\mathcal{B}$ for V, $\mathcal{C}$ for W

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots \vec{b}_n\}, \quad \mathcal{C} = \{\vec{c}_1, \vec{c}_2, \dots \vec{c}_m\}$$

$T: V \to W$ is represented by a matrix $A_{m \times n} = [T]_{\mathcal{B}}^{\mathcal{C}}$

$$\boxed{[T]_{\mathcal{B}}^{\mathcal{C}} = A = \left[[T\vec{b}_1]_{\mathcal{C}} \;\; [T\vec{b}_2]_{\mathcal{C}}, \dots \;\; [T\vec{b}_n]_{\mathcal{C}} \right]}$$

so for $\vec{x} \in V$, we can find $T(\vec{x})$ by
1) finding $[\vec{x}]_{\mathcal{B}}$,
2) finding $A[\vec{x}]_{\mathcal{B}} = [T(\vec{x})]_{\mathcal{C}}$ (matrix times vector)
3) finding $T(\vec{x})$

ex: Find $[T]_{\mathcal{E}}^{\mathcal{E}}$ where $T: \mathcal{P}^3 \to \mathcal{P}^3$ is given by $T(f(x)) = f'(x) + 4f(x)$

$\mathcal{E} = \mathcal{E}_3 = \{1, x, x^2, x^3\}$,

$$A = [T]_{\mathcal{E}}^{\mathcal{E}} = \left[[0+4]_{\mathcal{E}} \;\; [1+4x]_{\mathcal{E}} \;\; [2x+4x^2]_{\mathcal{E}} \;\; [3x^2+4x^3]_{\mathcal{E}} \right]$$

$$= \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$T(2x^3+5x+1) = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{pmatrix} 1 \\ 5 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 20 \\ 6 \\ 8 \end{pmatrix} = 9 + 20x + 6x^2 + 8x^3$$