Ex) Find $x^3 + 5x + 2 = f(x)$

in the basis $\mathcal{B} = \{1, x-3, \frac{(x-3)^2}{2}, \frac{(x-3)^3}{6}\}$

That is, find $[f(x)]_{\mathcal{B}}$, the col. vector representation of f, in basis $\mathcal{B}$.

$$\mathcal{B} = \{1, x-3, \tfrac{1}{2}x^2 - 3x + \tfrac{9}{2}, \tfrac{1}{6}x^3 - \tfrac{9x^2}{6} + \tfrac{27x}{6} - \tfrac{27}{6}\}$$

$$\left[\begin{array}{cccc|c} 1 & -3 & 9/2 & -9/2 & 2 \\ 0 & 1 & -3 & 9/2 & 5 \\ 0 & 0 & 1/2 & -3/2 & 0 \\ 0 & 0 & 0 & 1/6 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -9/2 & 9 & 17 \\ 0 & 1 & -3 & 9/2 & 5 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -9/2 & 17 \\ 0 & 1 & 0 & -9/2 & 5 \\ 0 & 0 & 1 & 0 & 18 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 44 \\ 0 & 1 & 0 & 0 & 32 \\ 0 & 0 & 1 & 0 & 18 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

so $\boxed{[f(x)]_{\mathcal{B}} = \begin{pmatrix} 44 \\ 32 \\ 18 \\ 6 \end{pmatrix}}$, $f(x) = 44 + 32(x-3) + 18\frac{(x-3)^2}{2} + 6\frac{(x-3)^3}{6}$

If you know calc II, that's the Taylor series for $f(x)$ at $x_0 = 3$,

$f(3) = 44$, $f'(3) = 3(3)^2 + 5 = 32$, $f''(3) = 18$, $f'''(3) = 6$

Ex) in $\mathbb{R}^2$, find $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ in the basis $\mathcal{B} = \{\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$

$$\left[\begin{array}{cc|c} -2 & 1 & 3 \\ 1 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ -2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 7 \end{array}\right]$$

so $\boxed{\left[\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}}$; $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 2\begin{pmatrix} -2 \\ 1 \end{pmatrix} + 7\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ✓

Change of basis

For a given basis $\mathcal{B}$, the matrix to row reduce is always the same, only the augment changes.

Note: row reduction move on A gives the same result as:
(same r.r. move on I)$\cdot A$

ex: $A = \begin{bmatrix} 7 & 8 & 9 \\ 2 & 1 & 3 \\ 9 & 3 & 4 \end{bmatrix} \xrightarrow{R2 \leftarrow R2 + 2R3} \begin{bmatrix} 7 & 8 & 9 \\ 20 & 7 & 11 \\ 9 & 3 & 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 7 & 8 & 9 \\ 2 & 1 & 3 \\ 9 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 \\ 20 & 7 & 11 \\ 9 & 3 & 4 \end{bmatrix} \checkmark$$

So if we row reduce I with all the same moves, just like for finding A^{-1}, we'll get a matrix that can do those moves (via multiplication) on any vector. It will be a change-of-basis matrix from $\mathcal{E}$ to $\mathcal{B}$. We call it $[I]_{\mathcal{E}}^{\mathcal{B}}$.

ex) $\mathcal{B} = \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

row reduce $\left[\begin{array}{cc|cc} -2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ -2 & 1 & 1 & 0 \end{array}\right]$

$\sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right]$ and $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ is the c.o.b.

$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$. $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} [\vec{x}]_{\mathcal{E}} = [\vec{x}]_{\mathcal{B}}$. $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = [I]_{\mathcal{E}}^{\mathcal{B}}$

For any two bases $\mathcal{B}$ and $\mathcal{C}$

we can find $[I]_{\mathcal{B}}^{\mathcal{C}}$. $\left(\begin{array}{l} \mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\} \\ \mathcal{C} = \{\vec{c}_1, \ldots, \vec{c}_n\} \end{array} \right)$

so $[I]_{\mathcal{B}}^{\mathcal{C}} [\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{C}}$

by $[I]_{\mathcal{B}}^{\mathcal{C}} = \Big[[\vec{b}_1]_{\mathcal{C}} \; [\vec{b}_2]_{\mathcal{C}} \; \ldots [\vec{b}_n]_{\mathcal{C}} \Big]$

columns are the basis vectors of $\mathcal{B}$, written as col. vectors in $\mathcal{C}$.

for our example $\rangle$ $[I]_{\mathcal{E}}^{\mathcal{B}} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \left[\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_{\mathcal{B}} \; \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]_{\mathcal{B}} \right]$

Note: $[I]_{\mathcal{B}}^{\mathcal{C}}$ is always square, $n \times n$.

$[I]_{\mathcal{B}}^{\mathcal{C}}$ is always invertible, and

$$\left([I]_{\mathcal{B}}^{\mathcal{C}}\right)^{-1} = [I]_{\mathcal{C}}^{\mathcal{B}}$$

example $\rangle$ $[I]_{\mathcal{B}}^{\mathcal{E}} = \left([I]_{\mathcal{E}}^{\mathcal{B}}\right)^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} = \left[\left[\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right]_{\mathcal{E}} \; \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_{\mathcal{E}} \right]$