ex) Is $\{x, x^2+1, 3x\}$ a basis for $\mathcal{P}^2$?

① $\boxed{\text{No.}}$ It has the right number of vectors (3) and it has polynomials of all the powers, but the third vector is a scalar times the first: $3x = 3(x)$. (lin. dep., so also does not span.)

② ex) Is $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \right\}$ a basis for $\mathbb{R}^3$?

$\boxed{\text{No}}$ Either notice that $\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

or row reduce $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(lin. dep., so also does not span) (Also: $\det A = 0$)

(Note ① & ② are same problem!)

ex) Is $\{x, x+3, 5x-1\}$ a basis for $\mathcal{P}^2$?

$\boxed{\text{No}}$: quick way: it has no x^2!

ex) Is $\{x^2+1, x^2+x, 3x, x-1\}$ a basis for $\mathcal{P}^2$?

$\boxed{\text{No}}$: it has 4 vectors but $\dim(\mathcal{P}^2) = 3$.

ex) Is $\{3x^2+2x, 5x-3\}$ a basis for $\mathcal{P}^2$?

$\boxed{\text{No}}$: it has only 2 vectors, so cannot span.

ex) Is $\{x^2+1, 1-x, x\}$ a basis for $\mathcal{P}^2$?

$\boxed{\text{Yes}}$ $\det \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = 0 - (-1) + 0 = 1 \neq 0$

(So spans & lin. indep.)

ex) Is $\{1, 2+3x, 4+5x+2x^2, 7x-x^2+4x^3\}$ a basis for $\mathcal{P}^3$?

$\boxed{\text{Yes}}$ $\det \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 3 & 5 & 7 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 24 \neq 0$. (So spans & lin. indep.)