Chp 3 cont

Def. The dimension of a vector space, $\dim(V)$ of V (or subspace S) is the number of vectors in any basis of V (or S).

ex) $\mathbb{R}^n$ has dimension n.
The standard basis for $\mathbb{R}^n$ is called $\mathcal{E} = \mathcal{E}(\mathbb{R}^n) = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$
where $\vec{e}_i$ = all zero components except one "1" in the i^{th} component.

$$\mathcal{E} \text{ for } \mathbb{R}^4 = \left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}$$

* $\mathcal{E}$ is ordered!

Note: $\mathbb{R}^n$ has many other bases (∞).

2-out-of-3 rule: if $\dim(V) = n$

set of $\boxed{n}$ vectors in V

set of vectors that $\boxed{\text{spans}}$ V

set of $\boxed{\text{lin. indep.}}$ vectors in V

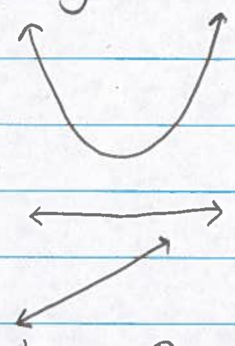
∴ any 2 of these implies the third!

- n lin. indep. vectors $\Rightarrow$ spans
- n vectors which span $\Rightarrow$ lin. indep.
- lin. indep. and spans $\Rightarrow$ exactly n vectors

A new vector space $\mathcal{P}^n$
is the set of all polynomials with degree at most n.

ex) $\mathcal{P}^2$ = all the polynomials with degree ≤ 2.

such as:
- x^2
- $3x^2 + 2$
- $\frac{1}{2}x^2 - 3x + 7$
- 5
- $x - 1$
- 0



always $= 0$, all x

$\mathcal{P}^2$ is a vector space: obeys all 8 axioms.

$\rightarrow \quad 3(3x^2+2) + (x-1) = 9x^2 + x + 5 \in \mathcal{P}^2$

$\rightarrow \quad 3x^2 + 2 + 0 = 3x^2 + 2$

ex) Is the set $\{x^2+3,\ x^2\}$ lin. indep?

Means: if $c_1(x^2+3) + c_2x^2 = 0$
then is $c_1 = c_2 = 0$ the only solution?

Solve:

Expand $\quad c_1x^2 + c_1 3 + c_2 x^2 = 0$

$\Rightarrow (c_1 + c_2)x^2 + c_1 3 = 0$

But this must be true for all x-values, including $x = 0$!

$\Rightarrow \quad c_1 3 = 0$

$\Rightarrow \quad \boxed{c_1 = 0}$

$\Rightarrow \quad (0 + c_2)x^2 + 0(3) = 0$

$\Rightarrow \quad c_2 x^2 = 0 \quad$ true for $x = 1$!

$\Rightarrow \quad \boxed{c_2 = 0} \quad \Rightarrow$ lin. indep.

Also, the set $\mathcal{E} = \{1, x, x^2\} \subseteq \mathcal{P}^2$ is lin. indep. Plus, it spans $\mathcal{P}^2$! (any deg 2 or less polynomial looks like $f(x) = c + bx + ax^2$, written $[f(x)]_{\mathcal{E}} = \begin{pmatrix} c \\ b \\ a \end{pmatrix}$)

"column vector for $\mathcal{E}$"

So it is a basis for $\mathcal{P}^2$, and it has 3 items (vectors), so $\mathcal{P}^2$ has dimension $= 3$.

In general $\mathcal{P}^n$ has $\dim(\mathcal{P}^n) = n+1$ and standard basis $\mathcal{E}(\mathcal{P}^n)$

This $\mathcal{E}$ is also ordered:

$$\mathcal{E} = \{1, x, x^2, x^3, \ldots, x^n\}.$$

An alternate basis for $\mathcal{P}^3$ is

$$\mathcal{B} = \{5, x^3+1, x^2, x+x^2\}$$

From now on, all our bases will have specific ordering.

Find $f(x) = 4x^3 + 2x$ as a lin. comb. of $\mathcal{B}$.

Method: in terms of standard basis $\mathcal{E}$ for $\mathcal{P}^3$

$$[f]_{\mathcal{E}} = f = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \end{pmatrix}, \quad \mathcal{B} = \left\{ \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

so solve (r.r.)

$$\left[\begin{array}{cccc|c} 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 5 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -4/5 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$f = -4/5\,(5) + 4\,(x^3+1) - 2\,(x^2) + 2\,(x+x^2). \quad \boxed{[f]_{\mathcal{B}} = \begin{pmatrix} -4/5 \\ 4 \\ -2 \\ 2 \end{pmatrix}}$$