ex) $S = \text{Span}\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$

is a subspace of $\mathbb{R}^2$.

But, that set of vectors is lin. dep.
(since $4 > 2$)
That means, some of those vectors can be made as lin. combs. of others, so the list is redundant: there is a smaller list whose span is S.

Def: a $\boxed{\text{basis}}$ $\mathcal{B}$ of a vector space V (or subspace) is a lin. indep. set of vectors $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ such that $\text{span}(\mathcal{B}) = V$.

To find a basis for S, row reduce the matrix of those column vectors,

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ in r.r.e.f.}$$

→ find the pivots, and then find the original columns in those positions (col 1 and 3)

→ Then $S = \boxed{\text{Span}\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix} \right\}}$ for $\mathcal{B} = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix} \right\}$

→ ($\mathcal{B}$ is a basis)

→ This S is also called the $\boxed{\text{column space}}$ $\text{col}(A)$ of $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

ex) Find the column space, as span of a basis, for $A = \begin{bmatrix} 3 & 0 & 6 & 0 & 1 & 2 \\ 4 & 0 & 8 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 7 & 14 \end{bmatrix}$

$$\text{r.r.} \sim \begin{bmatrix} 1 & 0 & 2 & 0 & 1/3 & 2/3 \\ 4 & 0 & 8 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 7 & 14 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 0 & 1/3 & 2/3 \\ 0 & 0 & 0 & 0 & -1/3 & -2/3 \\ 0 & 0 & 0 & 0 & 7 & 14 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 0 & 1/3 & 2/3 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 7 & 14 \end{bmatrix}$$

$$\sim \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccccc} \uparrow & & & & \uparrow & \end{array} \end{array}$$

$$\operatorname{col}(A) = \operatorname{Span}\left\{ \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix} \right\}$$

Note: if you know Calc 3; $\vec{n} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$

col(A) is a plane! $\vec{n} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 4 & 0 \\ 1 & 1 & 7 \end{vmatrix} = \langle 28, -21, -1 \rangle$

and $(0,0,0)$ is in the plane

so $28x - 21y - z = 0$ is an equation of the plane.

Also $\operatorname{col}(A) = \operatorname{Span}\left\{ \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix} \right\}$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{Bmatrix} x = 3t + s \\ y = 4t + s \\ z = 7s \end{Bmatrix} \right\}$$

is a parameterization of that plane.