So to decide lin. dep. or lin. indep,
we can always solve the vector equation
Unique solution $\vec{0}$ $\Rightarrow$ lin. indep.
∞ solution (free variables) $\Rightarrow$ lin. dep.

Shortcuts!

For $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ all vectors in $\mathbb{R}^m$
there are several shortcuts:

- if one of them (or more) is $\vec{x}_i = \vec{0}$, then lin. dep.

- if one of them (or more) is a scalar times another $\vec{x}_i = c\vec{x}_j$, then lin dep.
  [see previous example: $\vec{x}_3 = -2\vec{x}_1$]

- if one of them can be found as a lin. comb. of the others $\vec{x}_i = c_j \vec{x}_j + \dots + c_k \vec{x}_k$, then lin. dep.
  [here, the converse is also true.]

- if the number of vectors is larger than the number of components (dimension) of each ($n > m$), then lin. dep.

- if $n = m$ and $\det \left[\vec{x}_1 \; \vec{x}_2 \; \dots \; \vec{x}_n \right] = 0$
  then lin. dep.
  and if that det. $\neq 0$, then lin. indep.

ex) "$\{\ldots\}$" means "the set of"

1) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right\}$ $n = 4$, $m = 3$

→ $\boxed{\text{lin. dep.}}$ since $4 > 3$.

→ ∞ solutions to $A\vec{x} = \vec{0}$ if A is the matrix with these columns

→ at least one of these can be made as a lin. comb. of the others

2) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} \right\}$

→ $\boxed{\text{lin. indep.}}$ since $\det \overbrace{\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & -1 \end{bmatrix}}^{A} = -3 \neq 0$

→ only one solution $\vec{x} = \vec{0}$ to $A\vec{x} = \vec{0}$.

→ none of these can be made as a lin. comb. of the other two.

3) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \right\}$

→ $\boxed{\text{lin. indep.}}$ since if $\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$ is made as a lin. comb. of $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ that would mean

$\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = c \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, so $\left.\begin{matrix} 2 = c1 \\ 4 = c2 \end{matrix}\right\} c = 2$

$3 = c0 \rightarrow c = 2$ fails. $(3 \neq 0)$

→ two vectors are lin. dep. only when parallel; $\vec{x}_2 = c\vec{x}_1$.