Linear Dependence & Independence

$$\vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

→ a set of vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3, \ldots, \vec{x}_n$
is **linearly dependent**
when there exists a set of scalars $c_1, c_2, \ldots, c_n$
(which are <u>not all</u> equal to 0)
such that $c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n = \vec{0}$.

→ that same set of vectors is **linearly independent**
if there is no such set of scalars,
that is, $c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n = \vec{0}$
<u>only</u> when $c_i = 0$ for all $i = 1, \ldots, n$.

Ex) Are $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix}, \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix}$ lin. dep. or lin. indep.?

Solve $c_1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} + c_3\begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Same as solving this system:

$$\left.\begin{aligned} c_1 + 4c_2 - 2c_3 &= 0 \\ 2c_1 \quad\; - 4c_3 &= 0 \\ 3c_1 + 7c_2 - 6c_3 &= 0 \end{aligned}\right\} \text{homogeneous}$$

Same as solving: $A\vec{c} = \vec{0}$ with $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

$$\left[\begin{array}{ccc|c} \multicolumn{3}{c|}{\overbrace{\qquad\qquad}^{A}} & \\ 1 & 4 & -2 & 0 \\ 2 & 0 & -4 & 0 \\ 3 & 7 & -6 & 0 \end{array}\right]$$

Same as finding intersection of 3 homogeneous planes. Note $\vec{c} = \vec{0}$ is definitely a solution!

$\rightarrow$ Check that: $0\begin{pmatrix}1\\2\\3\end{pmatrix} + 0\begin{pmatrix}4\\0\\7\end{pmatrix} + 0\begin{pmatrix}-2\\-4\\-6\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ ✓

(This is always true: $A\vec{x} = \vec{0}$ always has at least one solution, $\vec{x} = \vec{0}$)

But: there could still be either 1 solution or ∞ solutions.

* Lin. dep. is another term for ∞ solutions to the "lin. comb. $= \vec{0}$" equation. Lin. indep. is a term for 1 unique solution, $\vec{0}$.

For practice, solve it!

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 2 & 0 & -4 & 0 \\ 3 & 7 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & -8 & 0 & 0 \\ 0 & -5 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -5 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} c_1 - 2c_3 = 0 \\ c_2 = 0 \\ c_3 = c_3 \text{ free} \end{cases} \quad \begin{cases} c_1 = 2c_3 \\ c_2 = 0 \\ c_3 = c_3 \end{cases}$$

General solution $\vec{c} = \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = c_3\begin{pmatrix}2\\0\\1\end{pmatrix}$

Note: homogeneous systems never have a non-$\vec{0}$ constant vector added to their solution.

Specific solutions: pick any c_3.

$$c_3 = 1 \Rightarrow \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix} \neq \begin{pmatrix}0\\0\\0\end{pmatrix}, \text{ so}$$

$\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\0\\7\end{pmatrix}, \begin{pmatrix}-2\\-4\\-6\end{pmatrix}$ are lin. dep.