Definition: (Hilbert + Dedekind)

An abstract plane geometry is:

I. (incidence)

D1 a set of points $\mathcal{P}=\{A,B,C,\dots\}$

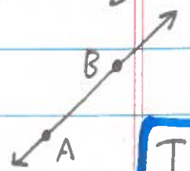
D2 a set of lines $\mathcal{L}=\{\ell,m,q,\dots\}$

*(all named points and lines are distinct, except in III.)

with: **S1** a relation $\mathcal{I}\subseteq\mathcal{P}\times\mathcal{L}$, called incidence

obeying: **I1** $\forall A,B\in\mathcal{P},\ \exists!\,\ell\in\mathcal{L}$ s.t. $(A,\ell),(B,\ell)\in\mathcal{I}$.

(any two points lie on exactly one line; and so we equate $\ell=\overleftrightarrow{AB}=\{C\in\mathcal{P}\mid(C,\ell)\in\mathcal{I}\}$)



I2 $\forall \ell\in\mathcal{L},\ \exists A,B\in\mathcal{P}$ s.t. $(A,\ell),(B,\ell)\in\mathcal{I}$.

I3 $\exists\, \ell,m\in\mathcal{L}$ s.t. $\ell\neq m$. (gives "2D" plane.)

II. (order)

with: **S2** a relation $\mathcal{B}\subseteq\mathcal{P}\times\mathcal{P}\times\mathcal{P}$, called betweenness.

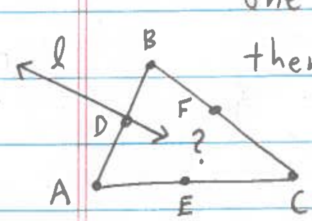
→ $(A,B,C)\in\mathcal{B}$ is stated: 'B is between A and C.

obeying **O1** $\forall A,B,C\in\mathcal{P}$, $(A,B,C)\in\mathcal{B}\Rightarrow(C,B,A)\in\mathcal{B}$, and all three points are on the same line $\ell=\overleftrightarrow{AC}$.

O2 $\forall A,C\in\mathcal{P},\ \exists B\in\mathcal{P}$ s.t. $(A,B,C)\in\mathcal{B}$. (gives ∞ points)

O3 $\forall A,B,C$ on line ℓ, exactly one point is between the others.

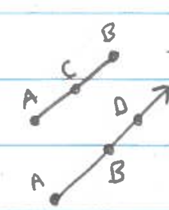
O4 $\forall A,B,C\in\mathcal{P},\ \ell\in\mathcal{L}$, if A,B,C are not all on one line, and ℓ does not contain any of them, then: if ℓ contains D with $(A,D,B)\in\mathcal{B}$ then ℓ must contain E with $(A,E,C)\in\mathcal{B}$ or contain F with $(B,F,C)\in\mathcal{B}$.



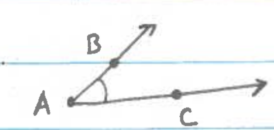
[definition break]

→ segment $\overline{AB}=\{A,B\}\cup\{C\mid(A,C,B)\in\mathcal{B}\}$

→ ray $\overrightarrow{AB}=\overline{AB}\cup\{D\mid(A,B,D)\in\mathcal{B}\}$

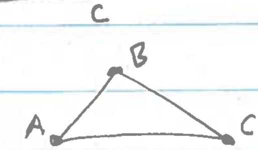


→ angle $\angle BAC=\overrightarrow{AB}\cup\overrightarrow{AC}$



→ triangle $\triangle ABC=\overline{AB}\cup\overline{BC}\cup\overline{AC}$

(often we include the interior)



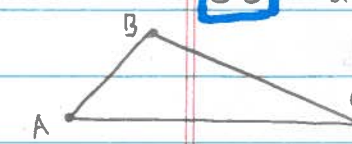
→ $\mathcal{S}\subseteq\mathcal{P}$ is convex means $A,B\in\mathcal{S}\Rightarrow\overline{AB}\subseteq\mathcal{S}$.

III. (congruence)

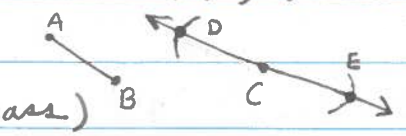
with: **S3** an equivalence relation on segments denoted $\overline{AB}\cong\overline{CD}$

S4 an equivalence relation on angles denoted $\angle ABC\cong\angle DEF$

S5 an equivalence relation on triangles denoted $\triangle ABC\cong\triangle DEF$ where order matters: 2 triangles are congruent when there exists matching between the two ordered lists of sets of 3 points, and $\triangle ABC\cong\triangle DEF$ in that order means $\overline{AB}\cong\overline{DE}$, $\overline{BC}\cong\overline{EF}$, $\overline{AC}\cong\overline{DF}$ and $\angle BAC\cong\angle EDF$, $\angle ABC\cong\angle DEF$, $\angle BCA\cong\angle EFD$.

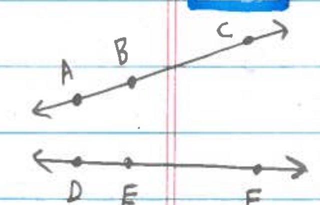


obeying **C1** $\forall\ \overline{AB}$ and $(C,\ell)\in\mathcal{I}$, $\exists!\ D,E$ on ℓ s.t. $(D,C,E)\in\mathcal{B}$ and $\overline{AB}\cong\overline{CD}\cong\overline{CE}$. (copy segment with compass)

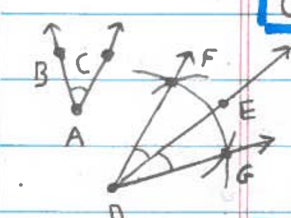


C2 Segment congruence is transitive: $\overline{AB}\cong\overline{CD}$ and $\overline{CD}\cong\overline{EF}\Rightarrow\overline{AB}\cong\overline{EF}$.

C3 Segment congruence is additive: $\forall A,B,C,D,E,F\in\mathcal{P}$, If $\overline{AB}\cong\overline{DE}$ and $\overline{BC}\cong\overline{EF}$ and $(A,B,C)\in\mathcal{B}$ on line ℓ and $(D,E,F)\in\mathcal{B}$ on line m, then $\overline{AC}\cong\overline{DF}$.



C4 $\forall\ \angle BAC$ and $\overrightarrow{DE}$, $\exists!\ \overrightarrow{DF}$ and $\overrightarrow{DG}$ s.t. $\overleftrightarrow{FG}$ contains one point of $\overleftrightarrow{DE}$ between F and G, (we say F and G are on opposite sides of $\overleftrightarrow{DE}$) and s.t. $\angle EDF\cong\angle EDG\cong\angle BAC$. (copy angle with compass)

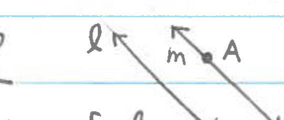


C5 Angle congruence is transitive.

C6 $\forall A,B,C,D,E,F\in\mathcal{P}$, If $\overline{AB}\cong\overline{DE}$, $\angle ABC\cong\angle DEF$, and $\overline{BC}\cong\overline{EF}$ then $\triangle ABC\cong\triangle DEF$. (SAS axiom).

IV. (parallels)

P1 $\forall$ line ℓ and point A not on ℓ, $\exists!\ m\in\mathcal{L}$ s.t. m contains A and m contains no points of ℓ.



V. (Dedekind continuity)

DC1 $\forall$ line ℓ partitioned into $\ell=S_1\cup S_2$ convex sets, $\exists!\ A$ on ℓ s.t. $S_i=\overrightarrow{AB}$ and $S_j=\overrightarrow{AC}-\{A\}$ with $(B,A,C)\in\mathcal{B}$, where $\{i,j\}=\{1,2\}$.