Equivalence relations, partitions, and Stirling's Δ

Def. A relation R on a set A is an <u>equivalence</u> relation when it is all three:

- symmetric, $(x,y) \in R \Rightarrow (y,x) \in R$
- transitive, $(x,y), (y,z) \in R \Rightarrow (x,z) \in R$
- reflexive, $(x,x) \in R$; for all $x, y, z \in A$.

Examples: $A = \{5, 0, 1, 2\}$

1) $R = \{(5,5), (0,0), (1,1), (2,2)\}$

reflexive ✓, symmetric, (vacuously) transitive ✓

2) $R = \{(5,5), (0,0), (1,1), (2,2), (1,2),$
$(2,1), (2,5), (5,2), (1,5), (5,1)\}$

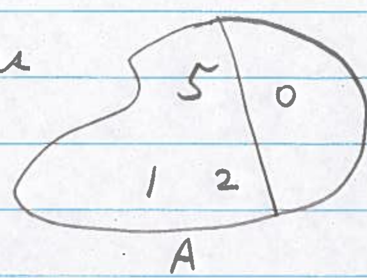
check all 3 ✓

Def. A <u>partition</u> of a set A is a collection of subsets of A: $U_1 \subseteq A$, $U_2 \subseteq A$, ...
such that:

- all are non-empty
- no two overlap
  (all intersections are empty
- the union of all U_i equals A.

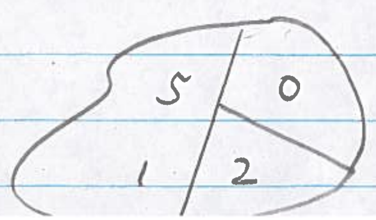
Examples

1)



two parts $U_1 = \{0\}$
$U_2 = \{5, 1, 2\}$

2)



three parts $U_1 = \{0\}$
$U_2 = \{5, 1\}$
$U_3 = \{2\}$

Theorem: for any set A, the equivalence relations on A are in one-to-one correspondence with the partitions of A (in bijection, so counted by the same numbers for finite A)

Proof: For a partition on A, create the relation $R \subseteq A \times A$ by including all the pairs in $U_i \times U_i$ for each part of the partition, (and no other pairs).

example: use (1), the two part partition above to make (2), the second example equivalence relation.

Check that R is always an equivalence relation:
- reflexive, since $(x,x) \in R$ since x is in some U_i for all $x \in A$
- symmetric, since if $(x,y) \in R$ then x and $y \in U_i$ for some U_i, so $(y,x) \in R$.
- transitive, since if $(x,y),(y,z) \in R$ then $x,y,z \in U_i$, so $(y,z) \in R$.

For an equivalence relation, the inverse construction is to partition A by creating the parts such that if $(x,y) \in R$ then x and y are in the same part. Check!

Stirlings Δ:

		row totals
$n=1$	1	1
$n=2$	1 1	2
$n=3$	1 3 1	5
$n=4$	1 7 6 1	15
*	+ 2x + 3x + 4x	
$n=5$	1 15 25 10 1	52
$n=6$	1 31 90 65 15 1	
	$\left\{ {6 \atop 1} \right\}$ $\left\{ {6 \atop 2} \right\}$ $\left\{ {6 \atop 3} \right\}$ $\left\{ {6 \atop 4} \right\}$ $\left\{ {6 \atop 5} \right\}$ $\left\{ {6 \atop 6} \right\}$	

* add above left to k times above right to find $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$

→ the number $\left\{ {n \atop k} \right\}$ is the number of partitions of [n] (n items) into k parts.

→ the row sums are the total number of partitions of [n], so also the total number of equivalence classes on [n].