Examples:

1) How many ways can 15 wrestlers (same class) place in the 2021 Olympics: gold, silver, bronze?

$$_{15}P_3 = 15 \cdot 14 \cdot 13 = 2{,}730$$

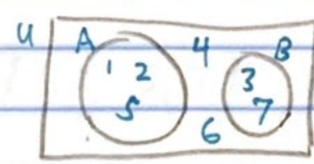
2) How many ways can 40 students fill up the front row of 5 seats (in order)?

$$_{40}P_5 = 40 \cdot 39 \cdot 38 \cdot 37 \cdot 36 = 78{,}960{,}960$$

Next we return to Addition and Subtraction.

In terms of sets:

Ex: $A = \{1, 2, 5\}$ $B = \{3, 7\}$, $U = \{1, \ldots, 7\}$



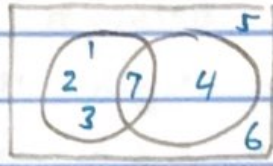
When $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B| = 5$

When U is the univeral set, $|\bar{A}| = |U| - |A| = 4$

When $A \cap B = \emptyset$, $|\overline{A \cup B}| = |U| - |A| - |B| = 2$

When there is overlap we have to avoid over counting!

Ex $A = \{1, 2, 3, 7\}$ $B = \{4, 7\}$



$$|A \cup B| = |A - B| + |A \cap B| + |B - A| = 3 + 1 + 1 = \boxed{5}$$

or, shortcut:

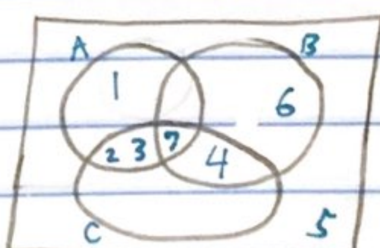
$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= \underbrace{4 + 2}\ \underbrace{- 1} = \boxed{5}$$

the shared stuff, $\{7\}$, gets counted twice: once in A, once in B, so we subtract it!

And $|\overline{A \cup B}| = |U| - |A| - |B| + |A \cap B| = 7 - 4 - 2 + 1 = \boxed{2}$

For 3 or more sets we can alternately add and subtract the overlaps to get the total:

Ex $A = \{1, 2, 3, 7\}$ $B = \{6, 7, 4\}$ $C = \{2, 3, 4, 7\}$



$\{7\}$ gets counted every time

$\{7\}$ gets subtracted 3 times

Add back in the very center! $\{7\}$

$$|A \cup B \cup C| = \underbrace{|A| + |B| + |C|} - \underbrace{|A \cap B| - |B \cap C| - |A \cap C|} + |A \cap B \cap C|$$
$$= 4 + 3 + 4 - 1 - 2 - 3 + 1 = \boxed{6}$$

and $|\overline{A \cup B \cup C}| = |U| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$

$= 7 - 6 = \boxed{1}$

Example

How many PIN's with 5 digits

but:
- no repeated digits
- First digit cannot be 0
- Third digit cannot be 2
- Fifth digit cannot be 5

Idea: let U be all the 5 digit PINS with no repeated digits

$|U| = {}_{10}P_5$

let A be PIN's with first digit 0.
let B be PINs with third digit 2.
let C be PINs with fifth digit 5.

Then legal PINs are $\overline{A \cup B \cup C}$.