Combinatorics. Review for Test 2

o.g.f. = ordinary generating function,
must be given as a function in closed form (no infinite sums or "...")

e.g.f. = exponential generating function,
must be given as a function in closed form (no infinite sums or "...")

zero is an even number

Don't simplify the generating functions you find: after getting the closed form just leave them as found.

1.a) Given an e.g.f. $f(x) = 3e^x + 2x$, for a sequence a_n, find a closed formula for a_n.

$$= 3\sum \frac{x^n}{n!} + 2x$$

$$a_n = \begin{cases} 3, & n \neq 1 \\ 5, & n = 1 \end{cases}$$

b) Given an e.g.f. $f(x) = e^{3x} + 4$, for a sequence a_n, find a closed formula for a_n.

$$= \sum \frac{3^n x^n}{n!} + 4$$

$$a_n = \begin{cases} 3^n, & n > 0 \\ 3^n + 4, & n = 0 \end{cases}$$

$$= \begin{cases} 3^n & n > 0 \\ 5 & n = 0 \end{cases}$$

2.a) Find the o.g.f. for the number of ways to choose n donuts from a menu that offers two types: chocolate and plain. You must choose exactly 4 or 5 chocolates, and a nonzero even number of plain.

$$f(x) = (x^4 + x^5)(x^2 + x^4 + \cdots)$$

$$\boxed{= (x^4 + x^5)\left(\frac{1}{1-x^2} - 1\right)}$$

$$= (x^4 + x^5)\left(\frac{x^2}{1-x^2}\right)$$

b) Find the o.g.f. for the number of ways to choose n donuts from a menu that offers two types: chocolate and plain, where you must choose at least 2 chocolate and an odd number greater than 3 of plain.

$$(x^2 + x^3 + \cdots)(x^5 + x^7 + x^9 + \cdots)$$

$$\left(\frac{1}{1-x} - x - 1\right)\left(x^5(1 + x^2 + x^4 + \cdots)\right)$$

$$= \boxed{\left(\frac{1}{1-x} - x - 1\right)(x^5)\left(\frac{1}{1-x^2}\right)} = \left(\frac{x^2}{1-x}\right)\left(\frac{x^3}{1-x^2} - x^3\right)$$

c) Given an o.g.f. for a sequence a_n: $f(x) = \dfrac{5}{(1-x)^2} + 3 + 7x^2$, find a closed formula for a_n.

$$= 5\sum_{n=0}^{\infty} n x^{n-1} + 3 + 7x^2$$

$$= 5\sum_{n=1}^{\infty} n x^{n-1} + 3 + 7x^2$$

$$= 5\sum_{n=0}^{\infty} (n+1) x^{n} + 3 + 7x^2$$

$$a_n = \begin{cases} 5(n+1) + 3 & n = 0 \\ 5(n+1) + 7 & n = 2 \\ 5(n+1) & \text{otherwise } n \neq 0, 2 \end{cases}$$

$$a_n = \begin{cases} 8 & n = 0 \\ 22 & n = 2 \\ 5(n+1) & \text{other } n \neq 0, 2 \end{cases}$$

3.a) Find the e.g.f. for the number of ways to arrange a permutation of length n using the letters A, C, with repetition, where there are an odd number of A's, and exactly 3 or 7 C's.

$$\boxed{\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{x^3}{3!} + \frac{x^7}{7!}\right)}$$

b) Find the e.g.f. for the number of ways to arrange a permutation of length n using the letters A, C, T, with repetition, where there are at least 2 A's, an even number of C's, and any number of T's.

$$\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(\frac{e^x + e^{-x}}{2}\right)e^x$$

$$= \boxed{\left(e^x - x - 1\right)\left(\frac{e^x + e^{-x}}{2}\right)e^x}$$

4. Given the recurrence relation for a sequence $a_n = 3a_{n-3} + 7$; $n \geq 3$; $a_0 = 3$, $a_1 = 0$, $a_2 = 5$.

(a) Find a_3, a_4, and a_5.

$$a_3 = 3a_0 + 7 = 16$$

$$a_4 = 3a_1 + 7 = 7$$

$$a_5 = 3a_2 + 7 = 22$$

(b) Find the o.g.f. $f(x)$ for a_n.

$$\underbrace{\sum_{n=3}^{\infty} a_n x^n} = 3\sum_{n=3}^{\infty} a_{n-3} x^n + 7\sum_{n=3}^{\infty} x^n$$

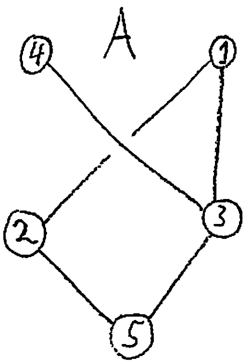
$$f - a_0 - a_1 x - a_2 x^2 = 3x^3 \underbrace{\sum_{n=3}^{\infty} a_{n-3} x^{n-3}}_{3x^3 F} + \frac{7}{1-x} - 7(1 + x + x^2)$$

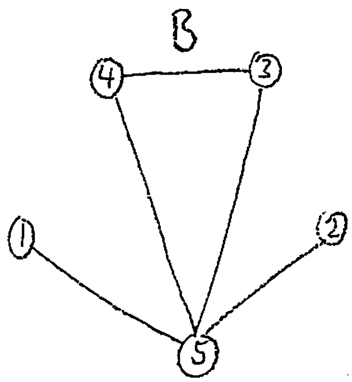
$$f - 3 - 5x^2 = 3x^3 f + \frac{7}{1-x} - 7(1 + x + x^2)$$

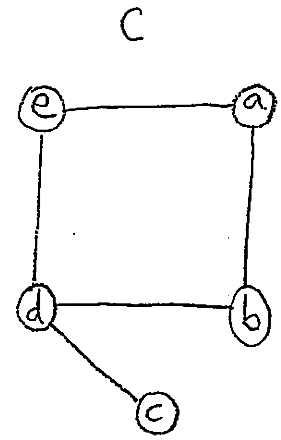
$$f(1 - 3x^3) = \frac{7}{1-x} - 7(1 + x + x^2) + 3 + 5x^2$$

$$\boxed{f = \frac{7}{(1-x)(1-3x^3)} - \frac{7(1+x+x^2)}{(1-3x^3)} + \frac{3+5x^2}{(1-3x^3)}}$$

5. For the graphs A,B,C pictured, answer the following:







a) Find the diameters:

diam(A)= 3

diam(B)= 2

b) Find the degree sequence deg.seq.(B) 4, 2, 2, 1, 1

c) Write in 'yes' or 'no.' In A, is 1,3,5,2,1,2,1

...a cycle? no ...is it a trail? no ...is it a walk? yes

d) Write in 'yes' or 'no.' In B, is 1,3,5,2,1

...a walk? no ...is it a trail? no ...is it a cycle? no

e) Find an isomorphism f from A to C. (List the inputs and outputs $f(\_) = \_$ for your isomorphism.)

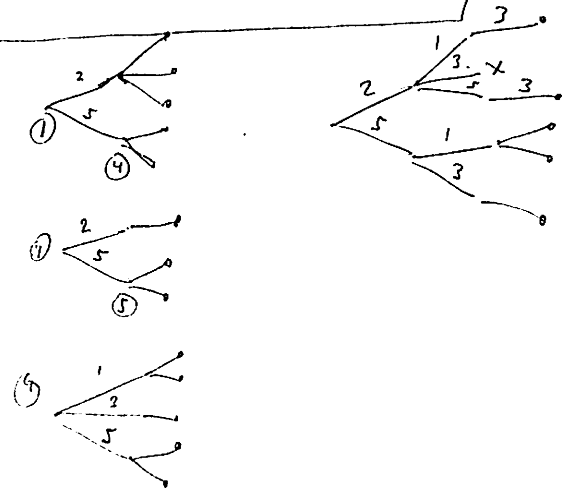
$f(4) = c$ $f(3) = d$

$f(1) = e$ $f(5) = b$

$f(2) = a$

6. Find the number of permutations φ of $\{1, 2, 3, 4, 5\}$ where $\varphi(1) \neq 2, 5$, $\varphi(4) \neq 1, 3, 5$, and $\varphi(5) \neq 2, 3$.

$$\boxed{5! - 7(4!) + 13(3!) - 5(2!)}$$



7.a) Use the o.g.f. for a_n which is $f(x) = \frac{x}{1-3x}$ to find the value of a_2.

$f'(x) = \frac{1-3x+3x}{(1-3x)^2} = \frac{1}{(1-3x)^2}$

$f''(x) = \frac{-1(2)(1-3x)(-3)}{(1-3x)^4}$

$f''(0)/2! = \frac{6}{2} = 3$

$= x \sum (3x)^n$

$= \sum 3^n x^{n+1}$

$= \sum_{n=1} 3^{n-1} x^n$

$\Rightarrow a_2 = 3^{2-1} = \boxed{3}$

b) Use the e.g.f. for a_n which is $f(x) = xe^{2x}$ to find the value of a_1.

$f'(x) = 1e^{2x} + xe^{2x}2$

$f'(0) = \boxed{1}$