Games

For a mathematician in the realm of college politics, capturing the throne of math department chair is often the only game in town. That game however, has only one winning move. Luckily, when it comes to mathematics research there are two kinds of game theory, and no reason not to play!

1. Classical games

Let's mention the first kind of game theory, which is the seriously much more important variety. (This will get it out of the way more quickly so we can spend more time on the fun kind.) John Nash won a Nobel prize in economics for proving this sort of game always has points of equilibrium, where if everyone is rational, no one unilaterally changes strategy. If that doesn't imply we can achieve world peace, at least it implies the existence of world stability, if everyone behaves rationally.

This is known as classical game theory, and it covers games where you cannot see all your opponents' decisions and options. Your information is incomplete, and you have to decide strategy based on the likelihoods and expected values of various possibilities; then everyone shows their cards at once. Of course we mean poker cards, and this also covers games like the prisoners' dilemma (snitch or deny), the battle of the gendered pastime options (opera vs football), the game of chicken (both players lose if neither folds), and iterated games where you can look at your opponents history to help guess their next move. Any game that includes a card deal or dice roll, from Monopoly or Risk or Clue, to Snakes and Ladders and Candyland, includes some blind strategy, even when you can logically solve puzzles along the way. That's because you have to decide how much risk to take (like betting on who wins Candyland), especially if you are playing for high stakes, like who has to wash the dishes later. Okay, that does all sound kind of fun, so maybe it will get a chapter section later, or some sidebars here.

2. The real stuff

The second type of game theory is known as combinatorial game theory. This is the sort we prefer, since our poker face is an open book. This includes all the games played on a board that both players can see entirely: chess, checkers, tic-tac-toe, go, and that dot game where you get to put your initial in any box you finish. There is no element of chance here, nor any opportunity for bluffing. Of course there are games that blend skill with blind moves, gambling, and luck. You can totally bet on Chess or Go, and we already mentioned using logic in Clue. Betting on chess is a bigger game than simply playing it though, and pushes things back into classic territory. So for now we will assume that there are two players, that the outcome is entirely determined by their choices, that all those choices are

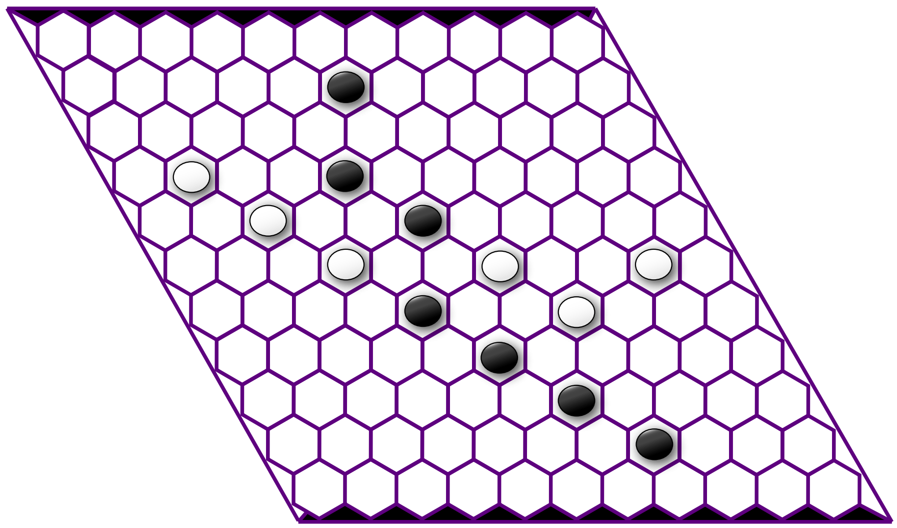


FIGURE 1. The standard hex board, 11 x 11. Here it is white's move but black has already secured a victory. White cannot block the path since there are two ways to connect each black stone.

public information as they take turns moving according to rules, and that the game will end in some victory or tie, depending on the rules.

The first game we will look at was invented by several folks, most famously perhaps by John Nash, although Piet Hein actually beat him to it independently. It is called Hex, and is played on a rhombus-shaped board filled with rows of hexagons. The sides of the rhombus are colored, alternating black and white. Two players take turns placing their mark or colored stone on any open hexagon, with the goal of filling in a string of connected hexagons that stretch between the two opposite sides of the entire rhombus of that color. Corner hexagons are conected to both sides they touch. Play continues until one player builds a connected path or until the entire board is filled...but at that point there will be a path!

3. Strategy stealing

Indeed the first thing to note about hex is that there is always a winner, no ties are possible. That's easiest to prove by seeing that defense and offense are the same strategy: blocking all your opponent's paths is the same thing as constructing your own successful path. It is impossible for both people to win, since a win for one will block the win for the other, and impossible for both to be blocked, for the same reason.

If one of the players must win, we can argue that there must be a strategy to achieve that win, a plan for where to move at each turn starting with the first move that player makes and giving a set of winning plays at each step based on the current state of the board. Whatever the other player does, that set of winning plays will never be empty for our happy strategist. This is provable in several ways, first by Zermelo. One way is to consider any winning position (say for black) and assume that the loser (white) did their very best to stop it. If that is true, it must have been achieved from a previous position in which black had some ways to win and white could not block them all. That was a winning position of its own, as when there is a checkmate coming in one move and no way to stop it. Then just before that, black made a move to create that position, and that move was a winning move. The position black saw at that point just before their winning move was "checkmate in 2" for black. This continues to the very first move: at every stage there must have been no way to stop the strategy, no way to reduce the winning moves to zero.

However, this argument doesn't say who has the winning strategy, just that one exists. The next thing to prove is that the person to play first can win: there exists a winning strategy for player one. This seems believable just due to the advantage of being one step ahead, but it is very nice to see the logical proof. First, since there is always a winner, lets assume for the sake of argument that player two, white, actually does have a winning strategy. This will turn out to be a proof by contradiction.

If player two has a winning strategy, then it must consist of a function that tells them where to move given any state of the board reached in the sequence of turns, starting with the position after black goes first. However, if based on the first black stone white knows where to go, then based on the first white stone black can likewise use the same function to choose its next move. In other words, black can steal the strategy from player 2, to become a new "player 2" (with an extra black stone somewhere on the board in an arbitrary spot, which black pretends is empty for the sake of calculating the next move). Now, black can continue to steal this strategy at every step. There are a few cases to consider. If the strategy, after some white move, indicates black take an empty space then we do so and continue. If the strategy indicates that black should take the position held by the arbitrary first black stone, then black has basically already made the required move just by having a stone there, and instead black picks another random empty space to take: it becomes the new arbitrary stone. Thus it can never be a disadvantage to have an extra stone on the board. If there are no empty spaces for black to occupy then the game already ended; but it must have ended with black as the winner on the previous move since black was using a winning strategy (as the new player 2). Now we have shown that assuming a winning strategy for player two really leads to a contradiction: both players are shown to have winning strategies simultaneously but we know that is impossible.

Thus our assumption must have been false to begin with: player 2 cannot have a winning plan. Since there is a winning plan, it must belong to player one.

The interesting part comes next: what is the winning strategy? It turns out to be quite complex, in fact it is different enough for each size of playing board that no one has been able to find a pattern. For a 2x2 game, it is easy enough. Black can choose any of the four hexagons, and then immediately has two ways to win so that white cannot block both. For a 3x3 game black wins by moving into the center of the board. As of this writing, the largest size board with a strategy all worked out is 10 by 10. The strategies for the 11 x 11 board and larger are still unknown.

Surrealism

The complexity of Hex is quite large; the size of the tree of possible games for the 11 x 11 board is nearly a googol. While studying the large sizes of games like Hex and Go, John Conway invented a new kind of number: the surreal numbers. He started by inventing a notation for 2-player games which describes the state of a game board (or whatever you use to play it) as a pair of sets L and R for the first and second players respectively. This is written $\{L \mid R\}$ and the two sets are the sets of all moves available to the two players; so the sets of all new game positions after making their move. Of course to use this to describe the state of a real game, we have to also mention whose move it is.
The surreal numbers are special versions of games where the game positions can be ordered, and for which all the positions in L are required to be less than those in R. It turns out that surreal numbers can correspond perfectly with all the integers, rational numbers, reals, ordinals, cardinals, and beyond. It's all a game.

4. Naughts and Crosses

Some of our proofs about Hex relied on the fact that exactly one of the players must always win. Some games have historically allowed ties to occur, like a tic-tac-toe game in which neither player 1 (X) nor player 2 (O) has the three in a row (vertical, horizontal, or diagonal) required. However, we can change the rule describing a win! We say that player 1 must achieve the three in a row to win, but that player 2 wins by either achieving 3 in a row or by simply preventing player 1 from doing so. What were tie games, are now called losses for player 1. Now the strategy stealing argument can proceed, since there are only two outcomes. But this time, it would not help player 1 to steal the strategy of blocking the achievement, since those tied games are now losses for him. Instead, we use the argument to prove that there is no way for player 2 to achieve 3 in a row. If there were, then player 1 could steal that strategy. As before, the extra X wouldn't get in the way of stealing the strategy, it can only help.

So since player 2 cannot achieve three in a row first, we have the two possible outcomes of a win by player 1 via achievement or a win by player 2 via prevention. (We assume that there are no mistakes made, that each player makes the best move that they can at each turn.) Thus the actual strategies for the players will be asymmetrical. Each gameboard must be examined to decide who has a winning strategy. For instance, on the classic 3x3 gameboard for tic-tac-toe player 2 has a winning strategy! No matter how player 1 starts out, player 2 can prevent the achievement of three in a row. This is pretty well known, but if you haven't ever tried to figure it out do so now before reading the next paragraph.

If player one puts their first X in a corner, player 2 should put O in the middle. If player 1 goes first in the middle, player 2 should go in any corner. If player 1 goes on an edge square (not a corner) then player 2 can choose an adjacent corner or the center. After that first strategic response, player 2 will always be able to block player 1's attempts to achieve three in a row.

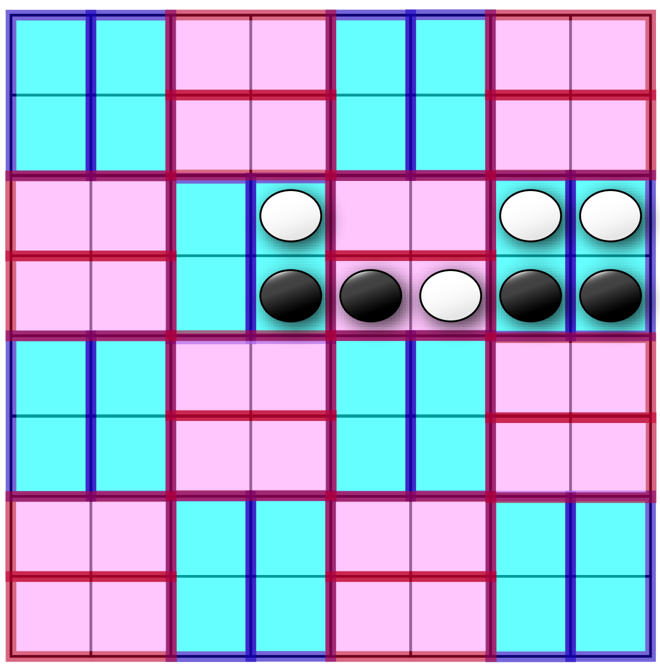
A key point is that the blocking strategy for player 2 depends on whether a win is declared for player 2 upon achieving 3 in a row. If the game is truly weak in that player 2 can only win by blocking, not by achieving, then in fact player 2 cannot block: player 1 is a winner! However we can even things out again by not allowing the three diagonal squares to count as a win for X. If player 1 must achieve three in a straight row, then again player 2 can succeed by blocking, without needing to threaten any achievement of their own.

On a 4x4 board things get better for player 1. Choosing a first move in one of the 4 central squares will win: no matter where O goes next there will be an option for X that yields two X's either horizontally or vertically in the center, with open squares on both sides of that pair: O cannot block both ends and the three in a row is achieved. Note that in that case, even if the diagonal three is not allowed, then player 1 is a winner. Also, it doesn't matter whether player 2 is allowed to threaten a win by aiming for their own achievement—player 1 is victorious in 3 moves so there is no opportunity to do so. The real hero here is the three in a row squares, vertical or horizontal. We stop pretending that the players matter so much, and we say that the figure itself is a winner on the 4x4 board, and a loser on the 3x3.

This unlocks the study of lots of possibilities, together known as achievement games. They are similar to Hex, but have the distinction that player 1 is trying to achieve a configuration (or one of several configurations) on the tiled gameboard but player 2 is trying to block it. (This is the weak achievement game.) Of course that's true for Hex as well—it just has a very large set of winning configurations (any path) and the feature that blocking looks like winning. That symmetry led to the fact that player 1 always has a strategy. The general case of achievement is less easy to prove! There are known winners, that is configurations (or sets of configurations) of cells one of which can be achieved on given boards, and there are known losers: configurations (or sets of them) all of which can be blocked. Not much has been worked out in terms of general theory. Achievement strategies are very specific, and known blocking strategies are fairly simple.

For a gameboard that is an infinite grid of squares, and achievement goals that are single configurations of squares connected along sides (called polyominoes), we often allow a win for achieving any rotation (by some multiple of 90 degrees) or reflection (vertical or horizontal) of that configuration. The three squares in a connected row is an example, and since it wins on 4x4 it certainly wins on the infinite grid. Four squares in a row also wins. See if you can prove it, by finding the strategy! Five in a row loses. That means there is a strategy to block, and here it is.

A blocking strategy for player 2 against the five in a row polyomino is a function that works as follows: the entire board is subdivided into sets of 2 squares each, called dominoes. We say the board is *paved* with dominoes, and in a repeating pattern that we can easily extend forever: pairs of dominoes vertical and horizontal, in a checkerboard. Here is the pattern (we went back to using black and white stones for X and O, respectively):



The key is that any 5 in a row, vertical or horizontal, will have to contain at least one domino. This can be seen pretty quickly by making the attempt to choose 5 in a row that use 5 different dominoes: it can't be done. Now for the strategy: player 2 has an easy function for deciding where to go: for any black stone placed by player 1, player 2 puts a white stone in the other half of that domino, as seen in the above picture. Since any winning configuration contains at least one entire domino, any potential win is blocked by a white before it can be achieved.

Counting

Now that we have brought up polyominoes we may as well start counting them! The free polyominoes are the ones that can be rotated and reflected, or flipped over, so that the original and its reflection are considered equivalent, and only count as one. That's how it feels when they are physical models, like the nice wooden versions we have here. These free polyominoes are also referred to as square animals. There is only one monomino and only one domino. There are 2 free triominoes and 5 free tetrominoes. There are 35 free pentominoes and 108 free hexominoes.
Alternatively you can consider the one-sided polyominoes which are counted as two different polyominoes when not symmetric under mirror reflection (or flipping over, which is the same thing for these 2D configurations.) Famously there are 7 one-sided tetrominoes, the 7 puzzle pieces in the game of tetris. Four of those come in pairs, and so there are 5 free tetrominoes. Free or not the total grows quickly, but any formula for this sequence is quite elusive! Open question: find any formula which takes n and computes the number of polyominoes with n squares. The formula can be closed, recursive, asymptotic, or a generating function. Just as hard: counting the symmetric polyominoes, or the chiral (not equal to their mirror image) polyominoes, or many other interesting subsets. One way to make it easier is to insist that they are tree-shaped and relatively convex. Stepping from square to connected square on such a polyomino one would always have a unique path between any two squares, like on a tree. The convexity here is to require that the total perimeter of the polyomino equals the perimeter of the smallest rectangle that it fits inside of, which eliminates things like spiralling branches and lagoons of unused squares. The fixed version (stuck in an xy-plane is sequence A196593, totally solved, and the free version is sequence A204804, almost solved.

5. Snaky

The best weak achievement game is the one for which we know no general strategy, neither for the first nor second player, neither to achieve nor to block. This is Snaky. It is a hexomino with 4 in a row, plus two more attached as the smiling end of the snake. Snaky is the only free polyomino which we do not know either an achievement nor a blocking strategy. On a small enough board, 8x8 or smaller there is a known blocking strategy. On a 9x9 board there may be a way for player 1 to win. Figure 3 shows a situation in which player 1 can secure a winning position with their move: black to move now, and win in 2 more moves.

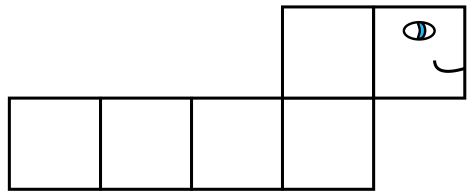


Figure 2. The traditional picture of Snaky.

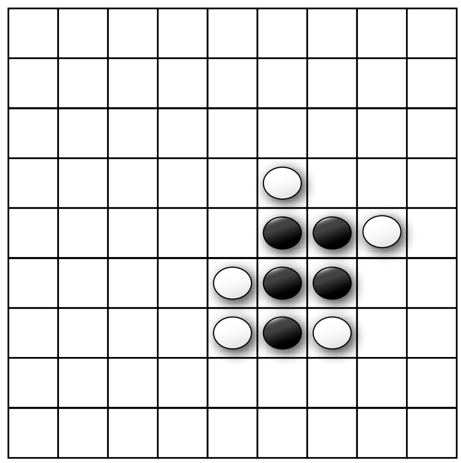


Figure 3. A checkmate for black, whose next move can seal the game: black to move and achieve a win in one more move.

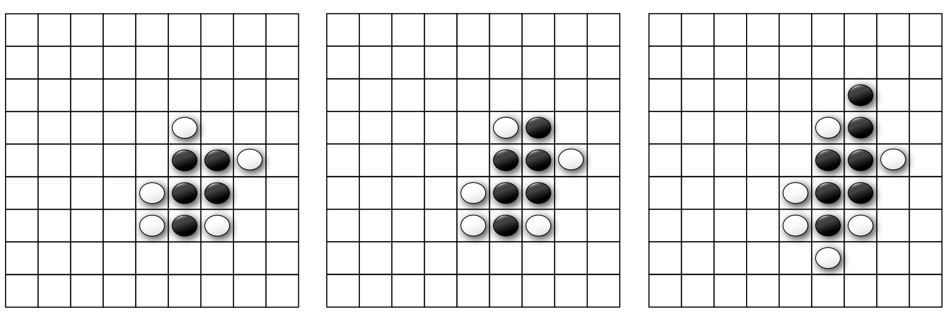


Figure 4. The move: it makes two ways to win, so victory is assured.

It has been shown that there is no paving strategy for defeating snaky, neither with dominoes nor with any partition of all the squares into pairs. It has also been shown that given one free move, snaky is a winner. An interesting question would be whether it is easier to find strategies for strong Snaky: to allow the second player to use the threat of creating their own snaky, like in classic tic tac toe.

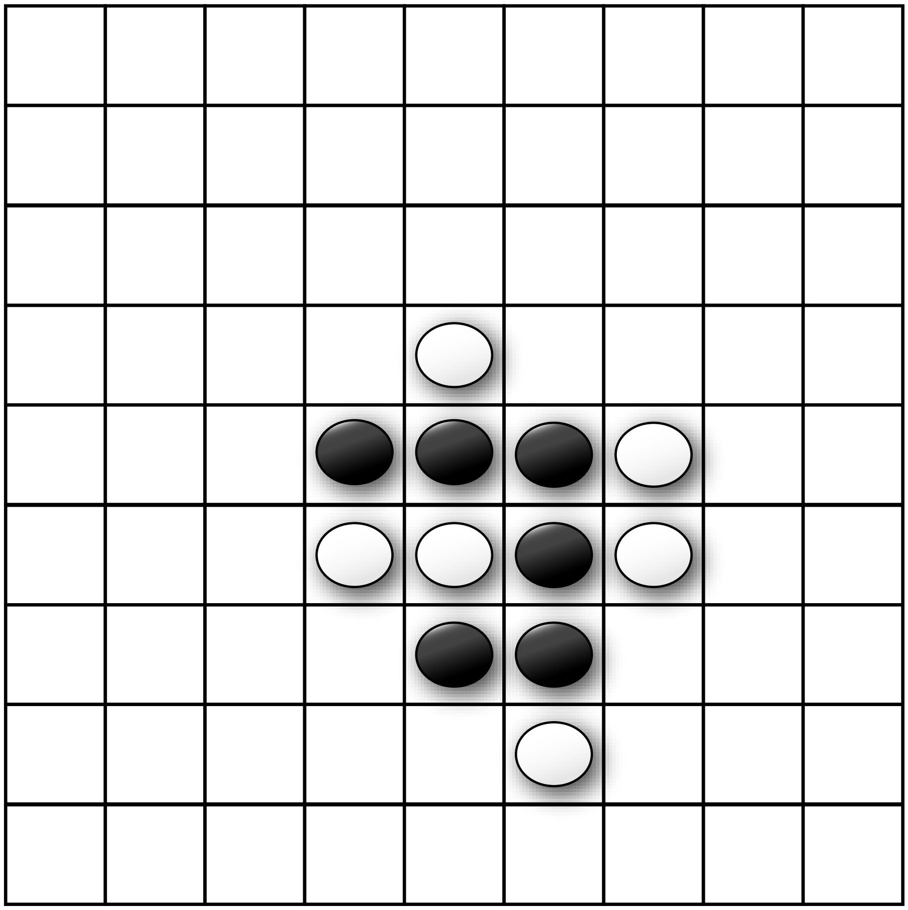


FIGURE 5. Puzzle this: black to move and win in two.