Ex: Find the number of integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 11$ where

Purchase 11 donuts, 4 types.

$$0 \le x_1 \le 4$$

no more than 4 of x_1

same as $2 \le x_2 \le 3$

$$2 \le x_2 < 4$$

at least 2 but less than 4 of x_2

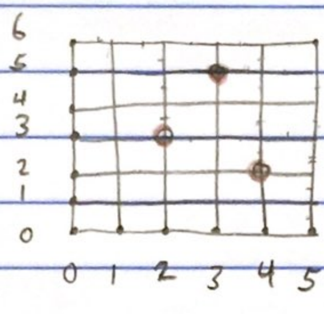
same as $3 \le x_3 \le 7$

$$2 < x_3 \le 7$$

more than 2 but no more than 7 of x_3

$$0 \le x_4 \le 11 .$$

Ex: Consider a grid graph with 5×6 edges. Find the number of shortest routes from $(0,6)$, upper left, to $(5,0)$, lower right, where we cannot use nodes: $(2,3)$, $(4,2)$, or $(3,5)$.



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1) $2 \le x_2$ and $2 < x_3$ $\Rightarrow$ we really only have $11 - 2 - 3 = 6$ in our total that can be distributed among the 4 variables

2) The upper limits are rules that cannot be broken. $x_4 \le 11$ isn't really a rule: it says x_4 can be anything up to the total. So only 3 rules really.

Keep in mind: $x_2 \ge 2$, $x_3 \ge 3$

x_1		? • • • • •
x_2	— —	? • •
x_3	— — —	? • • • • •
x_4		?

6

$$\binom{6+4-1}{4-1} - \underbrace{\binom{1+4-1}{4-1}}_{x_1 \ge 5} - \underbrace{\binom{4+4-1}{4-1}}_{x_2 \ge 4} - \underbrace{\binom{1+4-1}{4-1}}_{x_3 \ge 8}$$

$6-5$ $6-(4-2)$ $6-(8-3)$

$$+ \underbrace{0}_{\substack{x_1 \ge 5 \\ x_2 \ge 4}} + \underbrace{0}_{\substack{x_1 \ge 5 \\ x_3 \ge 8}} + \underbrace{0}_{\substack{x_2 \ge 4 \\ x_3 \ge 8}} - \underbrace{0}_{\substack{x_1 \ge 5 \\ x_2 \ge 4 \\ x_3 \ge 8}}$$

$$= \binom{9}{3} - \binom{4}{3} - \binom{7}{3} - \binom{4}{3}$$

$$= 84 - 4 - 35 - 4 = \boxed{41}$$

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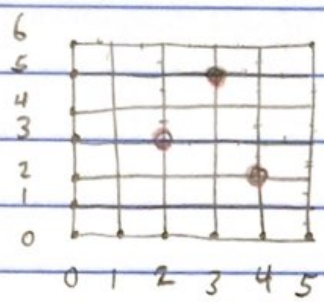

1) shortest walks all use 5 steps E and 6 steps S. (total 11 steps).
2) each forbidden node is a rule: breaking it means using it.
3) Notice: impossible to use $(2,3)$ and $(3,5)$ both on a shortest route: no backtracking!

$$\binom{11}{5} - \underbrace{\binom{5}{2}\binom{6}{3}}_{\text{use } (2,3)} - \underbrace{\binom{4}{3}\binom{7}{2}}_{\text{use } (3,5)} - \underbrace{\binom{8}{4}\binom{3}{1}}_{\text{use } (4,2)}$$

$$+ \underbrace{0}_{\substack{\text{(can't use} \\ \text{both } (2,3) + (3,5))}} + \underbrace{\binom{4}{3}\binom{4}{1}\binom{3}{1}}_{\substack{\text{use both } (3,5) \\ \text{and } (4,2)}} + \underbrace{\binom{5}{2}\binom{3}{2}\binom{3}{1}}_{\substack{\text{use both } (2,3) \\ \text{and } (4,2)}}$$

$$- \ 0 \quad \text{(can't use all 3)}$$

$$= 462 - 10 \cdot 20 - 4 \cdot 21 - 70 \cdot 3 + 4 \cdot 4 \cdot 3 + 10 \cdot 3 \cdot 3$$

$$= \boxed{106}$$