Multisets

Ex: $X = \{a, b, b, g, f, f, f, g\}$
$= \{1\cdot a, 2\cdot b, 2\cdot g, 3\cdot f\}$

$Y = \{m, i, s, s, i, s, s, i, p, p, i\}$
$= \{1\cdot m, 4\cdot i, 4\cdot s, 2\cdot p\}$

→ multicombinations, or multi subsets, of size k.

Ex: size 3 multisubsets, or 3-multicombinations of X:

$\{a, b, g\}$	$\{a, b, b\}$	$\{g, b, b\}$	$\{f, b, b\}$
$\{a, b, f\}$	$\{a, f, f\}$	$\{g, f, f\}$	$\{b, f, f\}$
$\{a, g, f\}$		$\{f, f, f\}$	
$\{b, g, f\}$	$\{a, g, g\}$	$\{b, g, g\}$	$\{f, g, g\}$

Goal: count these without listing them!

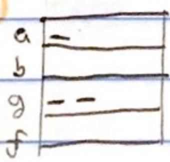
$\boxed{\text{Step 1}}$: 3-multicombinations of $Z = \{\infty \cdot a, \infty \cdot b, \infty \cdot g, \infty \cdot f\}$

$\{a,a,a\}$ ... etc.

$\{a, b, g\}$ ...

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Idea: 4 types a, b, g, f like 4 shelves
3 "blank elements" to assign to types



$= \{a, g, g\}$ $\Rightarrow$ the number of 3-multicombs of Z is $\binom{3+4-1}{4-1} = \binom{6}{3} = 20$.

Notation: $\left(\!\binom{4}{3}\!\right)$ // $\left(\!\binom{k}{n}\!\right) = \binom{k+n-1}{n-1}$

Finding 3-multicombinations of X ; counting.

$\boxed{\text{Step 2:}}$ Subtract the illegal multicombinations

- Cannot have more than 1 "a"
- Cannot have more than 2 "b"'s
- cannot have more than 2 "g"'s

Let U = all 3-multicombinations
A = those with 2 or more a's
B = those with 3 or more b's
G = those with 3 or more g's

Total =

$$|U| - |A| - |B| - |G| + |A\cap B| + |A\cap G| + |B\cap G| - |A\cap B\cap G|$$

$$= 20 - 4 - 1 - 1 + 0 + 0 + 0 - 0$$

{a,a,b} {a,a,f} {a,a,g}, {a,a,a}

= 14. ✓

→ multi permutations : How many (complete) permutations of $Y = \{1 \cdot m,\ 4 \cdot i,\ 4 \cdot s,\ 2 \cdot p\}$?

= How many anagrams of mississippi?

Idea: pretend the 4 i's are distinct

$m\, i_1\, ss\, i_2\, ss\, i_3\, pp\, i_4$ ... same for p, s...

Make the permutations, then realize we have a copy for every rearrangement of i_1, i_2, i_3, i_4: $4!$

Answer: $\boxed{\dfrac{11!}{\underset{\uparrow i}{4!}\ \underset{\uparrow s}{4!}\ \underset{\uparrow p}{2!}\ \underset{\uparrow m}{1!}}} = \boxed{34{,}650}$

A Bijection (one-to-one and onto correspondence.)

Note: $\binom{7}{3} = \frac{7!}{3!\,4!}$ = # anagrams of aaabbbb

So choosing an unordered subset of size 3 out of seven distinct elements... is in bijection with choosing an ordering of aaabbbb.

This makes sense! choosing an ordering of aaabbbb could be done by picking 3 out of 7 "blanks in order":

$$\underset{1}{\_}\ \underset{2}{a}\ \underset{3}{\_}\ \underset{4}{\_}\ \underset{5}{a}\ \underset{6}{a}\ \underset{7}{\_}$$ (fill in with b's)

$\Rightarrow$ babbaab.

bijection

one-to-one function

$\{2, 5, 6\}$

In general: # anagrams of mississippi

$$= \boxed{\binom{11}{4}\binom{7}{4}\binom{3}{2}\binom{1}{1}}$$

one spot remains for m

remaining 3 spots: 2 for p

out of remaining 7 spots choose 4 for s

choose 4 spots for i

$$= \frac{11!}{4!\,7!} \cdot \frac{7!}{4!\,3!} \cdot \frac{3!}{2!\,1!} \cdot \frac{1!}{1!\,0!} = \frac{11!}{4!\,4!\,2!\,1!}$$

Notation: $\binom{n}{j_1\ \ j_2\ \ \cdots\ \ j_k} = \dfrac{n!}{j_1!\, j_2! \cdots j_k!}$, $j_1 + j_2 + \cdots + j_k = n$

→ $\binom{7}{3} = \binom{7}{3\ \ 4}$ combinations are 2-letter multipermutations (in bijection with)

→ $\left(\!\binom{m}{k}\!\right) = \binom{k+m-1}{k} = \binom{k+m-1}{k\ \ \ m-1}$ multicombinations ↔ 2-letter multiperms.