Ch. 7 continued.

Recursion - recursive formulas - self-reference

$a_n = 2a_{n-1} + 5 \quad , n>0. \quad a_0 = 1$

n	a_n
0	1
1	$2a_0 + 5 = 7$
2	$2a_1 + 5 = 19$
3	$2a_2 + 5 = 43$

Idea: to find a shortcut to a_n we look for an o.g.f. $f(x)$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n .$$

Plan: Put both sides of the recursive equation into the sum $\sum_{n=0}^{\infty} \_\_\_ \, x^n$.

starting value for equation

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (2a_{n-1} + 5) x^n$$

since $n>0$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n - a_0 x^0 = \sum_{n=0}^{\infty} (2a_n + 5) x^{n+1}$$

$$\Rightarrow f(x) - 1 = 2x \sum_{n=0}^{\infty} a_n x^n + 5x \sum_{n=0}^{\infty} x^n$$

$$\Rightarrow f(x) - 1 = 2x f(x) + \frac{5x}{1-x}$$

$$\Rightarrow f(x)(1-2x) = \frac{5x}{1-x} + 1$$

$$\Rightarrow \boxed{f(x) = \frac{5x}{(1-x)(1-2x)} + \frac{1}{(1-2x)}}$$

Since $f(x)$ is an o.g.f.

- we can find $a_3 = f^{(3)}(0) / 3! = \boxed{43}$

- or, find the series for $f(x)$: series[$f(x)$] (wolfram)

$$= 1 + 7x + 19x^2 + \boxed{43}x^3 + \cdots$$

and a_n is the coefficient of x^n.

- or we can find the MacLaurin series for $f(x)$:

$$f(x) = \underbrace{\frac{5x}{(1-x)(1-2x)}} + \frac{1}{1-2x}$$

$$= \frac{A}{1-x} + \frac{B}{1-2x}$$

$$5x = A - 2Ax + B - Bx$$

$$\Rightarrow \begin{cases} A + B = 0 \\ -2A - B = 5 \end{cases}$$

$$\Rightarrow -A = +5$$

$$\Rightarrow \begin{cases} A = -5 \\ B = 5 \end{cases}$$

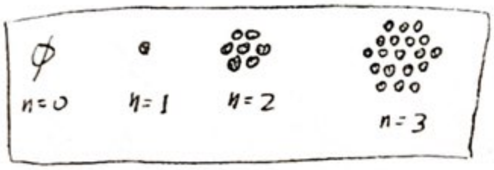
$$= \frac{-5}{1-x} + \frac{5}{1-2x} + \frac{1}{1-2x}$$

$$= -5 \sum_{n=0}^{\infty} x^n + 5 \sum_{n=0}^{\infty} (2x)^n + \sum_{n=0}^{\infty} (2x)^n$$

$$= \sum_{n=0}^{\infty} (-5 + 6(2^n)) x^n$$

$$\Rightarrow a_n = 6(2^n) - 5$$

$$a_3 = 6 \cdot 8 - 5 = \boxed{43}$$



$$\boxed{a_{n+1} = a_n + 6n, \quad n \geq 1} \qquad \boxed{\begin{array}{l} a_0 = 0 \\ a_1 = 1 \end{array}}$$

Since our recurrence is true for $n \geq 1$, we start with:

$$\boxed{\sum_{n=1}^{\infty} a_{n+1} x^n = \sum_{n=1}^{\infty} (a_n + 6n) x^n}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} a_{n+1} x^n - \underbrace{a_{0+1}}_{1} x^0 = \sum_{n=0}^{\infty} (a_n + 6n) x^n - \underbrace{(a_0 + 6 \cdot 0) x^0}_{0}$$

(mult. by x)

$$\Rightarrow \quad x \sum_{n=0}^{\infty} a_{n+1} x^n - x = x \sum_{n=0}^{\infty} (a_n + 6n) x^n$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} a_{n+1} x^{n+1} - x = x \sum_{n=0}^{\infty} a_n x^n + x \sum_{n=0}^{\infty} 6n x^n$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} a_n x^n - x = xf + x^2 \sum_{n=0}^{\infty} 6n x^{n-1}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} a_n x^n - \underbrace{a_0 x^0}_{0} - x = xf + x^2 \frac{6}{(1-x)^2}$$

$$\Rightarrow \quad f - x = xf + \frac{6x^2}{(1-x)^2}$$

$$\Rightarrow \quad f - xf = x + \frac{6x^2}{(1-x)^2}$$

$$\Rightarrow \quad f(1-x) = x + \frac{6x^2}{(1-x)^2}$$

$$\Rightarrow \quad f = \boxed{\frac{x}{1-x} + \frac{6x^2}{(1-x)^3}} \text{ o.g.f.}$$

Use:

$$\frac{x}{1-x} = \sum_{n=0}^{\infty} x^n - 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

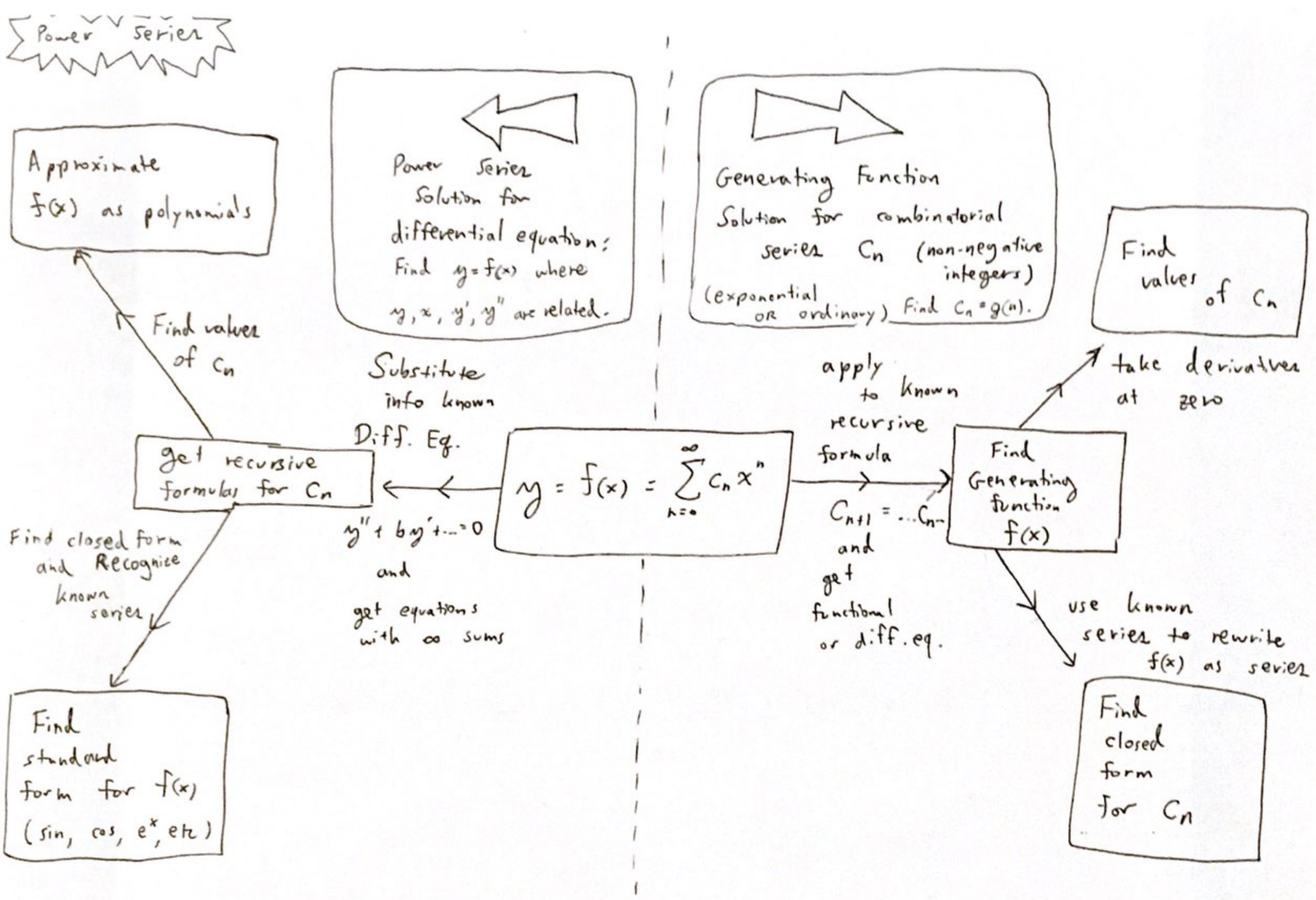
$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}$$

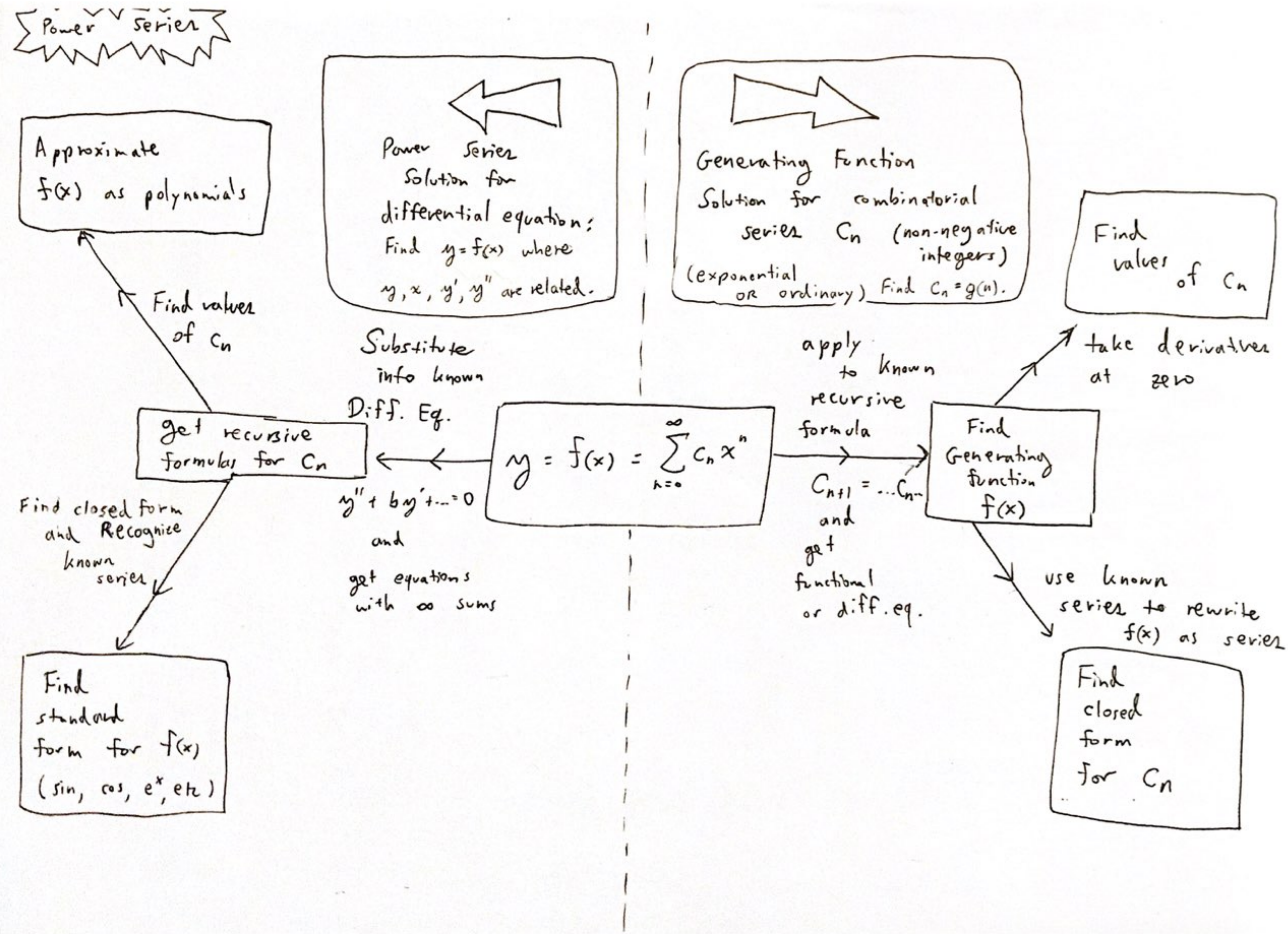
$$\frac{2}{(1-x)^3} = \sum_{n=0}^{\infty} n(n-1) x^{n-2}$$

Now $f = \sum_{n=0}^{\infty} x^n - 1 + 3x^2 \sum_{n=0}^{\infty} n(n-1) x^{n-2} = \sum_{n=0}^{\infty} (1 + 3n(n-1)) x^n - 1$

$$\boxed{a_n = \begin{cases} 1 + 3n(n-1), & n \geq 1 \\ 0, & n = 0 \end{cases}}$$

$\Uparrow$ closed formula


Power series
Power series Solution for differential equation: Find $y = f(x)$ where y, x, y', y'' are related.
Generating Function Solution for combinatorial series C_n (non-negative integers) (exponential or ordinary) Find $C_n = g(n)$.
Approximate $f(x)$ as polynomials
Find values of C_n
get recursive formulas for C_n
Substitute info known Diff. Eq.
$y'' + by' + \dots = 0$ and get equations with ∞ sums
$y = f(x) = \sum_{n=0}^{\infty} C_n x^n$
apply to known recursive formula
$C_{n+1} = \dots C_{n-}$ and get functional or diff. eq.
Find Generating function $f(x)$
Find values of C_n
take derivatives at zero
use known series to rewrite $f(x)$ as series
Find closed form for C_n
Find closed form and Recognize known series
Find standard form for $f(x)$ (sin, cos, e^x, etc)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take derivatives at zero
Substitute into known Diff. Eq.
apply to known recursive formula
get recursive formulas for C_n
$y = f(x) = \sum_{n=0}^{\infty} C_n x^n$
Find Generating function f(x)
$y'' + by' + \dots = 0$
and
get equations with ∞ sums
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