Example:

Let a_n = the number of PIN numbers (n-digit) using just the digits 1, 2, 3 with repetition, but no more than three "1"s.

$\boxed{n=1}$ $\quad \underline{1}, \underline{2}, \underline{3} \quad \boxed{a_1 = 3}$

$\boxed{n=2}$ $\quad \underline{1}\,\underline{1}, \underline{1}\,\underline{2}, \underline{2}\,\underline{1}, \ldots \quad \boxed{a_2 = 3 \cdot 3 = 9}$

$\boxed{n=3}$ $\quad \underline{1}\,\underline{1}\,\underline{1}, \underline{1}\,\underline{2}\,\underline{2}, \ldots \quad \boxed{a_3 = 3^3 = 27}$

$\boxed{n=4}$ $\quad \cancel{\underline{1}\,\underline{1}\,\underline{1}\,\underline{1}}, \underline{1}\,\underline{1}\,\underline{2}\,\underline{1}, \ldots \quad \boxed{a_4 = 3^4 - 1 = 80}$

$\boxed{n=5}$ $\quad \cancel{\underline{1}\,\underline{1}\,\underline{3}\,\underline{1}\,\underline{1}}, \underline{1}\,\underline{1}\,\underline{1}\,\underline{2}\,\underline{2}, \ldots$

Idea: the number of "1"s, "2"s, and "3"s in a PIN always add up to n. So let those be exponents of x again! $f(x)$ will have a factor for each of the three. But, if we count the ways to make a PIN (5 digit) by working with 3 "1"s and two "2"s it is found by $\frac{5!}{3!\,2!}$. In general, $\frac{n!}{j_1!\,j_2!\cdots j_k!}$.

So let $f(x) = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\right)$

choose this many "1"s

this many "2"s

this many "3"s

The coefficient of x^n will have the right denominators for each collection of digits, according to their repetition.

$n = 7$: $\dfrac{1}{2!\,4!\,1!} + \dfrac{1}{3!\,2!\,2!} + \cdots = $ coeff. of x^7 in expansion

2 "1"s, 4 "2"s, 1 "3" — 3 "1"s, 2 "2"s, 2 "3"s

$\underline{1}\ \underline{2}\ \underline{1}\ \underline{2}\ \underline{2}\ \underline{3}\ \underline{2}$

... But the coeff. just counts 1 for each collection of digits, so we need the 7! for the numerator. We need $n!$ in general.

So, $a_n = (\text{coefficient of } x^n \text{ in } f(x)) \cdot n!$

where $f(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right)\left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(1 + x + \frac{x^2}{2!} + \cdots\right)$

$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(e^x\right)\left(e^x\right)$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$= e^{2x}\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3\right)$

This is called the <u>exponential</u> <u>generating</u> <u>function</u> of a_n

Check: Find a_4 : (wolfram)

Series[e^(2x)*(1 + x + x^2/2 + x^3/6)]

$= 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{10x^4}{3} + \frac{29x^5}{15} + \cdots$

So $a_4 = \frac{10}{3} \cdot 4! = \boxed{80}$, $a_5 = \frac{29}{15} \cdot 5! = \boxed{232}$

☆

> **Def.** The exponential generating function (E.g.f.) of a_n is:
>
> $$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$
>
> as a MacLaurin power series.

Now compare this to the MacLaurin formula:

we get $\boxed{a_n = f^{(n)}(0)}$.

(since we already divided by $n!$)

Side-by-side

a_n	o.G.f. $f(x)$	E.g.f $g(x)$
→ Find one a_n	$a_n = f^{(n)}(0)/n!$	$a_n = g^{(n)}(0)$
→ Find all the a_n ... (series $[f(x)]$)	a_n = coeff. of x^n	a_n = (coeff. of x^n)($n!$)
→ Find function	$f(x) = \sum_{n=0}^{\infty} a_n x^n$	$g(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$
→ Use for........	best when unordered	best when ordering, permutations
→ Example	If $a_n = 1$, $f(x) = \frac{1}{1-x}$	If $a_n = n!$, $g(x) = \frac{1}{1-x}$
→ Example	If $a_n = 2^n$, $f(x) = \frac{1}{1-2x}$	If $a_n = 2^n$, $g(x) = e^{2x}$

Possibly: recapture a closed-form formula for a_n from the generating function.

Ex. Let a_n = number of multipermutations length n from the multisubset $\{\infty \cdot P, 1 \cdot Q, \infty \cdot R\}$ such that: there is always 1 "Q" and an even # of "R"'s. (0 is even).

$n = 0$: none $\rightarrow 0 = a_0$

$n = 1$: $\underline{Q}$ $\rightarrow 1 = a_1$

$n = 2$: $\underline{PQ}$, $\underline{QP}$ $\rightarrow 2 = a_2$

$n = 3$: $\underline{QPP}$, $\underline{PQP}$, $\underline{PPQ}$, $\underline{QRR}$, $\underline{RQR}$, $\underline{RRQ}$ $\rightarrow 6 = a_3$

$$g(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)(x)\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)$$

$$= e^x (x)\left(\tfrac{1}{2}(e^x + e^{-x})\right)$$

$$= \tfrac{1}{2} x e^{2x} + \tfrac{1}{2} x$$

$$= \tfrac{1}{2} x \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} + \tfrac{1}{2} x$$

$$= \tfrac{1}{2} \sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n!} + \tfrac{1}{2} x$$

Shift index: n to n-1

$$= \tfrac{1}{2} \sum_{n=1}^{\infty} \frac{2^{n-1} x^n}{(n-1)!} + \tfrac{1}{2} x$$

Multiply by n/n

$$= \tfrac{1}{2} \sum_{n=0}^{\infty} \frac{n 2^{n-1} x^n}{n(n-1)!} + \tfrac{1}{2} x$$

$$= \tfrac{1}{2} \sum_{n=0}^{\infty} \left(n 2^{n-1}\right) \frac{x^n}{n!} + \tfrac{1}{2} \frac{x^1}{1!}$$

$$a_n = \begin{cases} \tfrac{1}{2} n 2^{n-1}, & n \neq 1 \\ \tfrac{1}{2} n 2^{n-1} + \tfrac{1}{2}, & n = 1 \end{cases}$$

$$\boxed{a_n = \begin{cases} n 2^{n-2}, & n \neq 1 \\ 1, & n = 1 \end{cases}}$$

check: $a_3 = 3 \cdot 2^{3-2} = \boxed{6}$