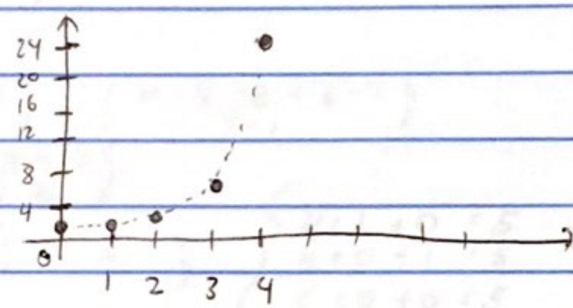
Combinatorial Sequences

→ a_n = function of n, $n = (0), 1, 2, 3, \dots$

1) Example: $a_n = n!$

n	a_n
0	1
1	1
2	2
3	6
4	24

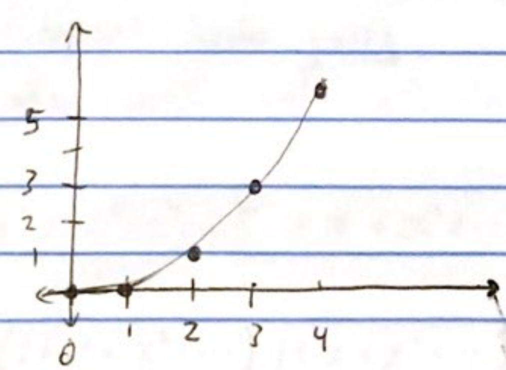


This grows faster than any exponential : superexponential growth.

Here $a_n = n! = \left|\{ \varphi : [n] \to [n] \}\right|$

2) Example: $a_n = \binom{n}{2} = \frac{n(n-1)}{2} = \frac{1}{2}n^2 - \frac{1}{2}n$

n	a_n
0	0
1	0
2	1
3	3
4	6
5	10



Quadratic growth.

Here $a_n = \binom{n}{2} = \left|\{ S \subset [n] \mid |S| = 2 \}\right|$

Combinatorial sequences count something!

Example:

Let a_n be the number of non-negative integer solutions to

$$x_1 + x_2 + x_3 = n$$

with $x_1 \geq 4$

$$\Rightarrow \quad a_n = \binom{n-4+3-1}{3-1} = \binom{n-2}{2}$$

Ex: $n=5$, $a_5 = \binom{3}{2} = 3$; $\begin{cases} 4+1+0=5 \\ 4+0+1=5 \\ 5+0+0=5 \end{cases}$

Or, alternative: Notice the fact that when you have the same base, exponents are added: $x^4 x^1 = x^5$.

Idea: $a_5 = 3$ is the coefficient of x^5 in the expansion of...

$$f(x) = (x^4 + x^5 + \dots)(x^0 + x^1 + \dots)(x^0 + x^1 + \dots)$$

... and the "..." means keep going,

so a_n is the coeff. of x^n in $f(x)$.

$$f(x) = (x^4 + x^5 + x^6 + \dots)(1 + x + x^2 + \dots)(1 + x + x^2 + \dots)$$

$$= x^4(1 + x + x^2 + \dots)(1 + x + x^2 + \dots)(1 + x + x^2 + \dots)$$

$$= x^4\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x}\right)$$

$$= \frac{x^4}{(1-x)^3}$$

The ordinary generating function of a_n

The ordinary generating function, o.g.f, G.f,

$f(x)$ of a_n is the

function where a_n is the coeff. of x^n.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$ as a MacLaurin power series.

This gives us a new way to calculate a_n.

Compare to MacLaurin formula: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$$\Rightarrow \quad a_n = f^{(n)}(0)/n!$$

Ex: find a_5 where $f(x) = \frac{x^4}{(1-x)^3}$.

1) 5^{th} derivative $f^{(5)}(x)$
2) plug in $x=0$
3) divide by $5!$

1) wolframalpha.com d^5/dx^5 ((x^4)/((1-x)^3))

$$= 360(2x^2+4x+1)/(x-1)^8$$

2] ; $x=0 \Rightarrow 360$

3) $360/5! = \boxed{3}$