Q: Is there a complete graph invariant?

Answer: Yes, but noone has found anything simple. The graph isomorphism problem is a potential career!

Graphs show connections, which allow moves.

Def: A walk on a graph G is any sequence of vertices for which successive vertices in the sequence are (connected by) edges. We say the walk uses those edges.

Def: A trail is a walk that never uses an edge more than once.

Def: A path is a trail that never uses a vertex more than once (never revisits a vertex.).

Def: A cycle is (almost) a path: it revisits only one vertex, which is both the starting and ending vertex.

Def: A closed walk has the same vertex as its starting and ending vertex.

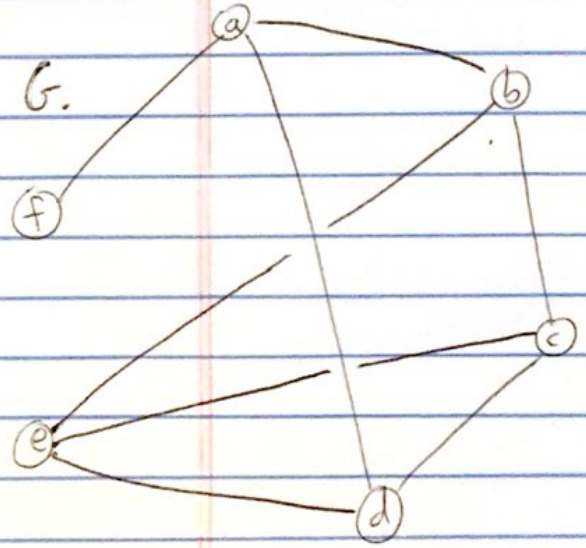
Def: A closed trail has the same vertex as its starting and ending vertex.

Def: The length of a walk (trail, path, cycle) is the number of edges, or one less than the length of the sequence of vertices.

Def: The (minimum path) distance for two vertices $x, y \in V(G)$ is the minimum length of any path starting at x and ending at y.

$$d(x,y) = \min_{P} \{ \text{length}(p) \mid p \text{ is a path from } x \text{ to } y \}.$$

Def: The diameter $\text{diam}(G)$ of a graph is the longest distance $d(x,y)$ between any two vertices of G.



(b, c, a, f) not a walk (not anything)

(c, b, c, d, a, f) ✓ walk. Not a trail, uses $\{b, c\}$ twice.

(e, b, c, e, d, a) ✓ trail. Not a path, uses e twice.

(f, a, b, e, d) ✓ path. Not a cycle.

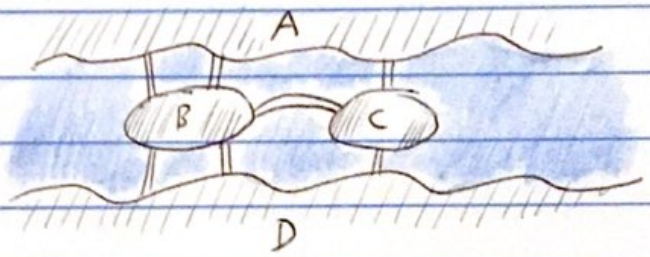
$d(f, c) = 3$, (f, a, b, c) or (f, a, d, c)

$\text{diam}(G) = 3$

* diameter is a graph invariant.

Def. An Eulerian trail is a trail which uses every edge of G.

Konigsberg bridges:

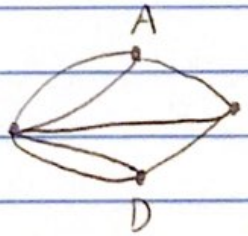


Q. Is there an Eulerian trail?

Recall: deg.

deg A = deg D = 3
deg C = 3
deg B = 5

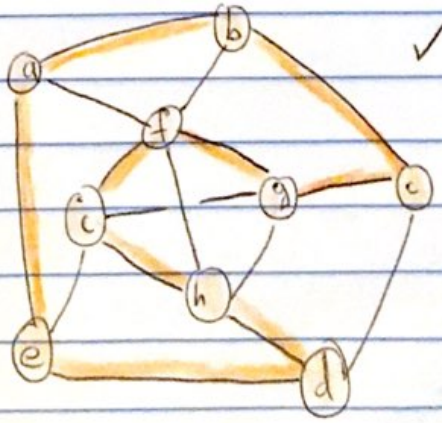
$G =$



} multigraph (duplicate edges)

A. If there is an Eulerian trail then every vertex must have even degree; except for the starting and ending vertex (for non-closed trail).
The converse is true as well!

Def. A Hamiltonian path is a path which uses (visits) every vertex of G.
A Hamiltonian cycle is a cycle that uses every vertex of G.



✓ (a, b, c, g, f, i, h, d, e, a)

Is there an Eulerian trail?