11 Graph Theory

Def. of Simple Graph (graph)

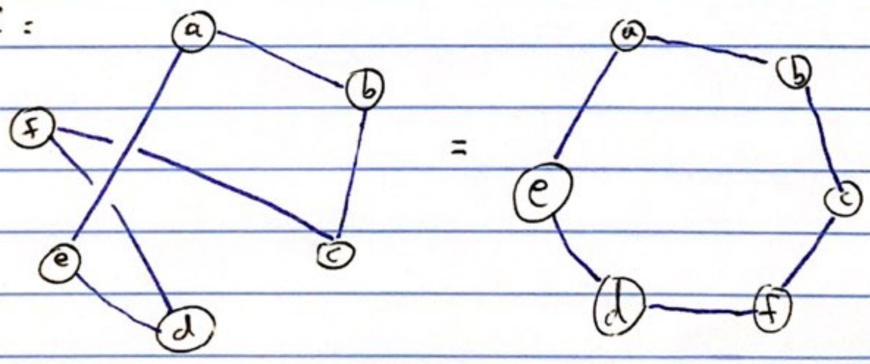
$$G = (V, E)$$

where $V = V(G)$ = finite set (of nodes or vertices)

and $E = E(G)$ = finite set of unordered pairs of distinct vertices from V, called edges.

Ex: $G = (\{a,b,c,d,e,f\} = V, \{\{a,b\},\{c,b\},\{d,f\},\{e,d\},\{a,e\},\{c,f\}\} = E)$

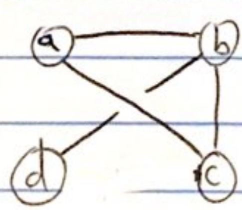
Drawing: G:

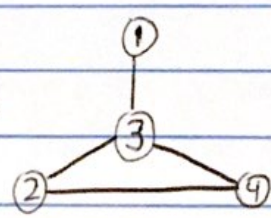


Def. Two graphs are isomorphic $G \cong H$ means: there is a bijection $f: V(G) \to V(H)$ such that f and f^{-1} take edges to edges:

$$\{f(x), f(y)\} \in E(H) \iff \{x, y\} \in E(G).$$

Ex: Show $G \cong H$

G = 

H = 

Proof:

x	$f(x)$
a	2
b	3
c	4
d	1

check four edges

✓

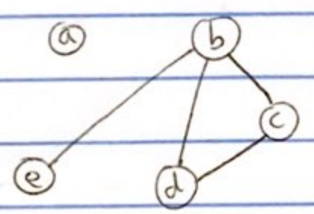
Some counting: If G has n vertices then the maximum number of edges is $\binom{n}{2}$. The complete graph $K_n = G$ has all the edges.

For two graphs G, H with n vertices each, there are $n!$ possible bijections. (We could check them all to see if there is an isomorphism)

Shortcuts! If $G \not\cong H$, we might not have to check all $n!$ bijections. Instead we could find invariants.

Def. The degree of a vertex in G is the number of edges which include (touch) that vertex.

Ex:

G: 

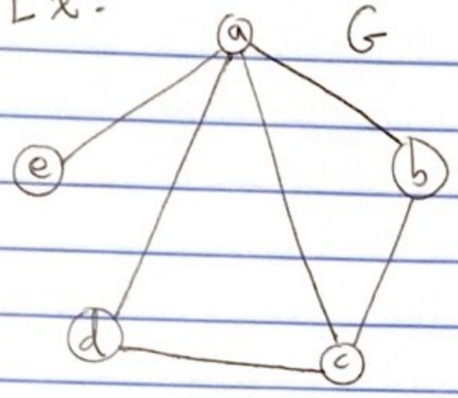
$\deg(a) = 0$
$\deg(b) = 3$
$\deg(c) = 2$
$\deg(d) = 2$
$\deg(e) = 1$

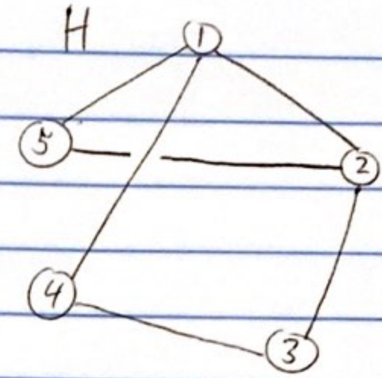
Def: The degree sequence of G is the list of degrees from largest to smallest.

$\text{degseq}(G) = (3,2,2,1,0)$

Theorem: If $G \cong H$ then $\text{degseq}(G) = \text{degseq}(H)$.
Thus degseq is a graph invariant.
Use to show that $G \not\cong H$ via contrapositive!
$\text{degseq}(G) \neq \text{degseq}(H) \Rightarrow G \not\cong H$.

Ex:

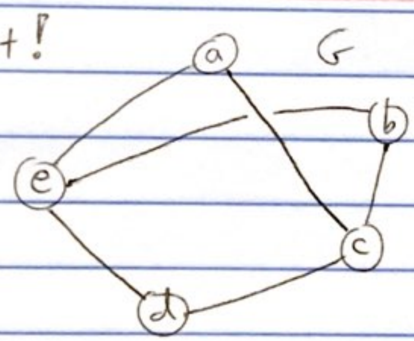


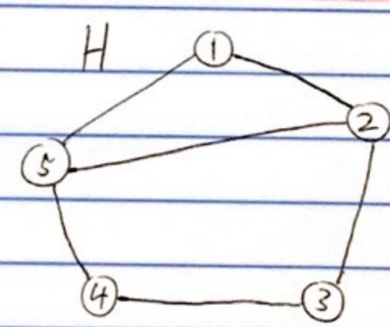


$\text{degseq}(G) = (4, 3, 2, 2, 1)$ $\text{degseq}(H) = (3, 3, 2, 2, 2)$

$\Rightarrow G \not\cong H$

But!





$\text{deg seq}(G) = (3, 3, 2, 2, 2)$ $\text{deg seq}(H) = (3, 3, 2, 2, 2)$

Q: Does $\text{degseq}(G) = \text{degseq}(H) \Rightarrow G \cong H$?

A: No, the converse of our theorem is not true.

Def. If an invariant "works both ways" and is described with "if and only if" then it is called a <u>complete</u> invariant.

Is there a complete graph invariant?