Def. A graph G is <u>connected</u> means there is a path (at least one) between every pair of vertices

Definition.

A graph G is <u>bipartite</u> means

that there exists a bipartition of the vertices such that every edge contains one vertex from each part.

That is; $G = (V, E)$ and

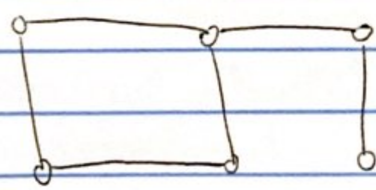
i.) there exists a way to split the set V into two non-empty, non-overlapping subsets

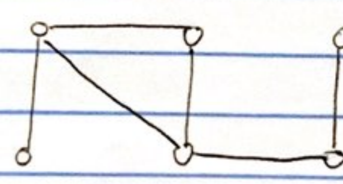
$$V = B \cup R$$

(blue vertices and red vertices).

ii) ... such that the edges $\{a, b\} \in E$ are always half red, half blue; one vertex from R and one from B.

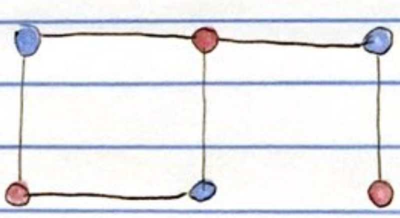
Ex:

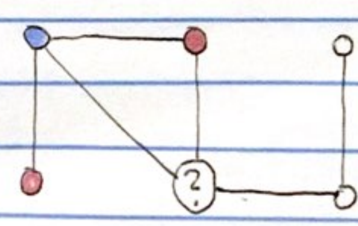
$G =$ 

$H =$ 

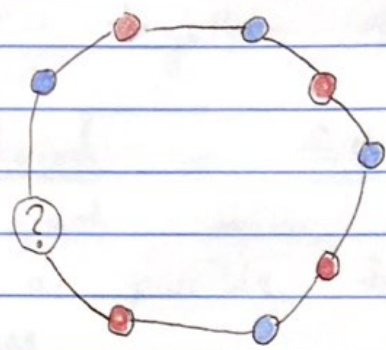
G is bipartite.

H is <u>not</u> bipartite.





Theorem: A graph G is bipartite
if and only if ($\Leftrightarrow$)
all cycles have <u>even</u> length.



$\Rightarrow$ So, no triangles! cycles must be at least length 4 in simple bipartite G.

Note: <u>bipartite</u> is a graph invariant property.

Theorems: (For Connected Graphs G)

1) (Recall) G has a closed Eulerian trail
iff ($\Leftrightarrow$) every vertex has even degree.
G has an open Eulerian trail
iff ($\Leftrightarrow$) there are exactly 2 vertices
with odd degree.

2) Hamiltonian paths & cycles.

A) Necessary conditions for Hamiltonian

$|R|$ = number <u>red</u>
$|B|$ = number <u>blue</u>

Theorem. If G is bipartite and
$|R| \neq |B|$ then ($\Rightarrow$)
there is <u>no</u> Hamiltonian cycle.

If G is bipartite and $\big||R| - |B|\big| \geq 2$
then there is <u>no</u> Hamiltonian path.

Definition: A <u>subgraph</u> of a graph G is a subset of the vertices of G, and a subset of the edges of G which together form a graph $H \subset G$.

Def.

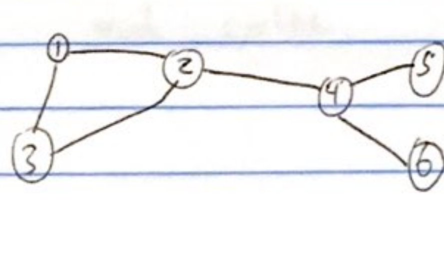
An <u>induced</u> or <u>full</u> subgraph is a subgraph which contains as many edges as possible from the original set of edges.

Def.

The <u>connected components</u> of G are the full subgraphs that are "islands" with no paths between their respective sets of vertices. They are the largest connected full subgraphs of G. If G is connected, then G has only one connected component.

Def.

A <u>bridge</u> of a graph G is an edge $\{a,b\}$ such that removing $\{a,b\}$ from the set E of edges gives us a subgraph with more connected components than G (we keep both a and b in the vertex set.)

$G =$ 

Bridges: $\{2,4\}$

Also $\{4,5\}$

$\{4,6\}$

Def: The order of a graph G, $n = \text{Ord}\,G$, is the number of vertices of G.

Notes: The number of connected components is a graph invariant. The number of bridges is a graph invariant. Order is a graph invariant. Etc!

Theorem: (another necessary condition)

If G has a bridge
then G has no H-cycle
(Hamiltonian cycle)

Theorem: Sufficient condition. (Ore condition)
Let n = Order of G, $n \geq 3$
If for all x, y vertices of G
we have $\deg(x) + \deg(y) \geq n$
then there is a Hamiltonian cycle.

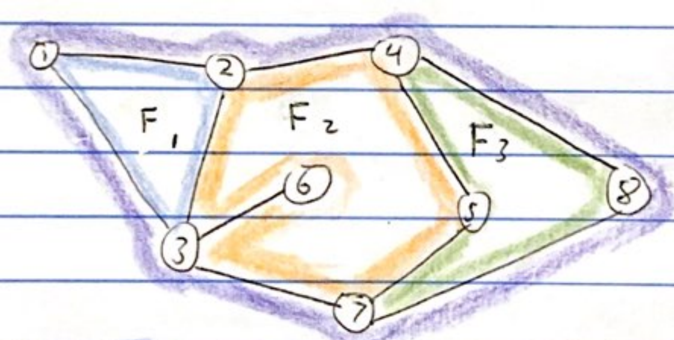
If for all x, y vertices of G
we have $\deg(x) + \deg(y) \geq n-1$
then there is a Hamiltonian path.

Open Question: It is unknown whether there are necessary and sufficient ($\Leftrightarrow$) conditions for Hamiltonian paths and cycles.

Def. A graph G is planar if it can be drawn on the plane $\mathbb{R}^2$ with no crossing (intersecting) edges.

Given a drawing of a planar graph G, we see $\mathbb{R}^2$ divided into regions (facets) $F = \{F_1, F_2, F_3 \dots\}$ each with a boundary made of edges; we count the number of sides of edges and say the region F_i has f_i "walls" or sides of edges.

$G =$



$\boxed{f_1 = 3}$ $\boxed{f_2 = 7}$ $\boxed{f_3 = 4}$ $\boxed{f_4 = 6}$

Theorems: 0) For a simple graph $f_i \geq 3$. Bipartite $\Rightarrow f_i \geq 4$

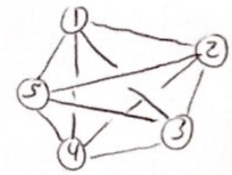
1) Euler: For planar $G = (V,E)$ with regions F:

$$|V| - |E| + |F| = 2$$

ex: $8 - 10 + 4 = 2$

2) $$\sum_{i=1}^{|F|} f_i = 2|E|$$ (sum of walls = twice edges)

ex: $3+7+4+6 = 2(10)$

1) Use Euler's formula to prove that K_5 = 

is not planar.

(10 edges)

→ Assume K_5 is planar, with $|F|$ regions.

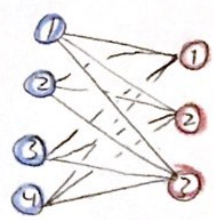
① $\Rightarrow |F| = 10 - 5 + 2 = 7$ (by Euler.)

② We have that $f_i \geq 3$ for each region. (simple graph)

③ $\Rightarrow \sum_{i=1}^{7} f_i \geq 3|R| = 21$ (each region has at least 3 walls, all 7 regions)

④ But: $\sum_{i=1}^{7} f_i = 2|E| = 20$ (by planarity)

⑤ $\Rightarrow 20 \geq 21$ ⨳ contradiction

2) Use Euler's formula to prove that $K_{4,3}$ = 

is not planar. (complete bipartite)

→ Assume $K_{4,3}$ is planar, with $|F|$ regions.

(12 edges)

① $\Rightarrow |F| = 12 - 7 + 2 = 7$ (by Euler)

② since bipartite, we have $f_i \geq 4$ for each region.

③ $\Rightarrow \sum_{i=1}^{7} f_i \geq 4(7) = 28$ (4 walls, 7 regions)

④ But: $\sum_{i=1}^{7} f_i = 2|E| = 24$ (by planarity)

⑤ $\Rightarrow 24 \geq 28$ ⨳