Homework #3 : Chapter 5

#6 $(3x-2y)^{18}$ coef of x^5y^{13}

$$(3x-2y)^{18} = (3x+(-2y))^{18} = \sum_{K=0}^{18} \binom{18}{K} (3x)^K (-2y)^{n-K}$$

x^5y^{13} : $5+13=18$ so $K=5$ and $n-K=13$

$$\text{coef. } \binom{18}{5}(3x)^5(-2y)^{18-5} = \binom{18}{5} 3^5 x^5 (-2)^{13} y^{13}$$

$$\binom{18}{5} 3^5 (-2)^{13} x^5 y^{13}$$

Thus, $\boxed{\text{the coefficient of } x^5y^{13} \text{ is } \binom{18}{5} 3^5 (-2)^{13}}$

$\boxed{\text{the coefficient of } x^8y^9 \text{ is } 0}$

#8 $\quad 2^n \stackrel{?}{=} \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k}$

$$2^n = (3-1)^n = (-1+3)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 3^{n-k}$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k}$$

Thus, $\boxed{2^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k}}$

#9 $\quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} 10^k$

$$(-10+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-10)^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 10^k$$

$(-10+1)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 10^k$ and $(-10+1)^n = (-9)^n$

so, $\boxed{\sum_{k=0}^{n} (-1)^k \binom{n}{k} 10^k = (-9)^n}$

#15 $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

let differenciate both sides with respect to x

$$n(x+y)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k\, x^{k-1} y^{n-k}$$

let $x = -1$ and $y = 1$

so $n(0)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k (-1)^{k-1} 1^{n-k}$

$$\rightarrow \quad 0 = \sum_{k=0}^{n} k \binom{n}{k} (-1)^{k-1}$$

thus, $\boxed{\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} + \cdots + (-1)^{n} n \binom{n}{n} = 0}$

#40 coef. of $x_1^3 x_2^3 x_3 x_4^2$ of $(x_1 - x_2 + 2x_3 - 2x_4)^9$

is $\dfrac{9!}{3!\,3!\,1!\,2!} 1^3 (-1)^3 (2)^1 (-2)^2$

$$\boxed{= \frac{9!}{3!3!2!}(-8)}$$