MacLaurin : $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, x^n}{n!} = \frac{f(0)}{1} + \frac{f'(0)\, x^1}{1} + \frac{f''(0)\, x^2}{2!} + \dots$

$f(x) = \sin x$ Find MacLaurin series.

$$\begin{aligned}
f^{(0)}(x) &= \sin x & \xrightarrow{\boxed{x=0}} \quad f^{(0)}(0) &= 0 \\
f'(x) &= \cos(x) & f'(0) &= 1 \\
f''(x) &= -\sin x & f''(0) &= 0 \\
f'''(x) &= -\cos x & f'''(0) &= -1 \\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\
f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1 \\
f^{(6)}(x) &= -\sin x & f^{(6)}(0) &= 0 \qquad f^{(7)}(0) = -1 \\
&\vdots & &\vdots
\end{aligned}$$

$$f^{(n)}(0) = \begin{cases} 1 & n = 1, 5, 9, \dots \\ -1 & n = 3, 7, 11, \dots \end{cases}$$

$$\sin(x) = 0 + \frac{1\,x}{1} + 0 + \frac{-1\,x^3}{3!} + 0 + \frac{1\,x^5}{5!} + 0 + -1\,\frac{x^7}{7!} + \dots$$

powers are odd #s: $2n+1$ (factorials match powers).
coefficients are ± 1

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!}$$

Some More!

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)\,(x-a)^n}{n!}$$

→ good for approximating $f(x)$ near $x = a$.

Ex: approximate e^x near $x = 2$, to the $n = 3$ power of x. Use it to find $e^{2.1}$.

$$e^x \approx \underbrace{\frac{e^2}{0!} + \frac{e^2(x-2)}{1!}}_{\text{this much is linearization from 3.10}} + \frac{e^2(x-2)^2}{2!} + \frac{e^2(x-2)^3}{3!}$$

$$e^{2.1} \approx 7.389 + 7.389(.1) + \frac{7.389(.1)^2}{2} + \frac{7.389(.1)^3}{6}$$

$$= 8.166$$

Ex: Find the $n=4$ term in the Taylor series for $f(x) = 5x + 2^x$ around $a = 1$.

$f(x) \quad 5x + 2^x$

$f'(x) = 5 + 2^x \ln 2$

$f''(x) = 2^x \ln 2 \ln 2$

$f'''(x) = 2^x \ln 2 \ln 2 \ln 2$

$f^{(4)}(x) = 2^x (\ln 2)^4$

$$\text{term} = \frac{f^{(4)}(a)(x-a)^4}{4!} = \frac{2^1(\ln 2)^4(x-1)^4}{4!}$$