11.9, 11.10 Functions that power series converge to.

Easy example:

$$\sum_{n=1}^{\infty} x^n \quad \text{converges to ?}$$

and where?

Ratio test:
$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| = \lim_{n\to\infty} |x| = |x|$$

so converges if $|x| < 1$.

test endpoints $1, -1$

both diverge by lim test for div.

Note: we already knew this, by geo series test since $\sum_{n=1}^{\infty} x^n$ converges to $\frac{x}{1-x}$ exactly when $|x| < 1$.

So the function $\boxed{f(x) = \frac{x}{1-x}}$ can be

written as $\boxed{\sum_{n=1}^{\infty} x^n} = x + x^2 + x^3 + x^4 + \dots$

whenever $|x| < 1$. (so x in $(-1, 1)$)

Or
$$\boxed{\sum_{n=0}^{\infty} x^n} = 1 + x + x^2 + x^3 + x^4 + \dots$$
$$= 1 + \frac{x}{1-x}$$
$$= \frac{1-x}{1-x} + \frac{x}{1-x} = \boxed{\frac{1}{1-x}}.$$

If we know that a power series converges to a function $f(x)$ with radius of convergence R, then so do $f'(x)$ and $\int f(x)\,dx$.

Other tricks: substitute or multiply by a factor.

Use: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

Ex: Find a power series that converges to:

(1) $\frac{1}{1+x} = \frac{1}{1-(-x)} = \boxed{\sum_{n=0}^{\infty} (-x)^n}$

(2) $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \boxed{\sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}}$

(3) $\frac{x^3}{1-x} = x^3\left(\frac{1}{1-x}\right) = x^3 \sum_{n=0}^{\infty} x^n = \boxed{\sum_{n=0}^{\infty} x^{n+3}}$

(4) $\ln(1+x) = \int \frac{1}{1+x}\,dx = \boxed{\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}}$

(5) $\tan^{-1} x = \int \frac{1}{1+x^2}\,dx = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$

Ex: What does $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = ?$

$= \sum_{n=0}^{\infty} \frac{(-1)^n\, 1^{2n+1}}{2n+1} = \tan^{-1} 1 = \boxed{\frac{\pi}{4}}$