Alternating series test

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad , \quad a_n > 0$$

monotone

if $\boxed{a_{n+1} \leq a_n}$ (decreasing)

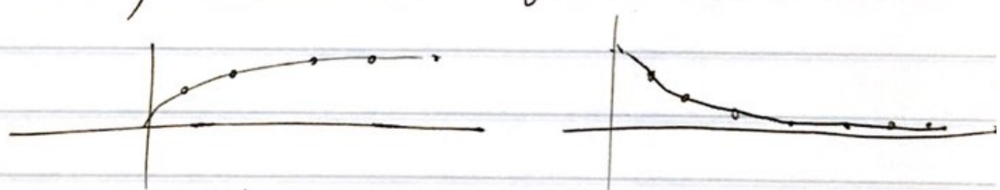
and $\boxed{\lim_{n\to\infty} a_n = 0}$

then converges.

Ex. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots$ converges

$= -\ln 2$

Ex. $\sum_{n=1}^{\infty} \left(\frac{\pi}{2} - \tan^{-1}(n)\right)(-1)^n$ converges.



Absolute vs Conditional convergence

[both $\sum |a_n|$ and $\sum a_n$ converge] $\to$ $\boxed{\text{Only } \sum a_n \text{ converges}}$

$\to$ Note: if $\sum |a_n|$ converge then $\sum a_n$ does too.

Ratio Test

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, inconclusive.

n	1	2	3	4	...	100	200	10 000	1000 000
$\left(1+\frac{1}{n}\right)^n$	2	2.25	2.37	2.44		2.705	2.712	2.718	2.718282

e

Ex:

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{3^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}}\right| = \frac{(n+1)^3 \cdot 3^n}{3^{n+1} n^3}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^3 3^n}{3^{n+1} n^3}$$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{3n^3} = \lim_{n\to\infty} \frac{n^3+3n^2+3n+1}{3n^3}$$

$$= \frac{1}{3} < 1$$

$$\boxed{\text{Equation: } \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}$$

so converges.

Ex $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\boxed{\begin{aligned}\text{Use} \quad n! &= n\cdot(n-1)(n-2)\cdots 3\cdot 2\cdot 1\\ &= n(n-1)!\end{aligned}}$$

$$\lim_{n\to\infty} \frac{(n+1)^{(n+1)}}{(n+1)!}\cdot\frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^{n+1} n!}{(n+1)n!\, n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)(n+1)^n}{(n+1)\, n^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e > 1 \text{ diverges}$$

Root test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$$

then $\sum a_n$ is absolutely convergent.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$$

then $\sum a_n$ diverges

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$$ then inconclusive.

Ex: $$\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$$

$\lim = \frac{2}{3} < 1$

converges

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n}$$

$$= \frac{2}{3}$$