11.3 The integral test.

If $f(x)$ is continuous, positive, decreasing function on $[1, \infty)$

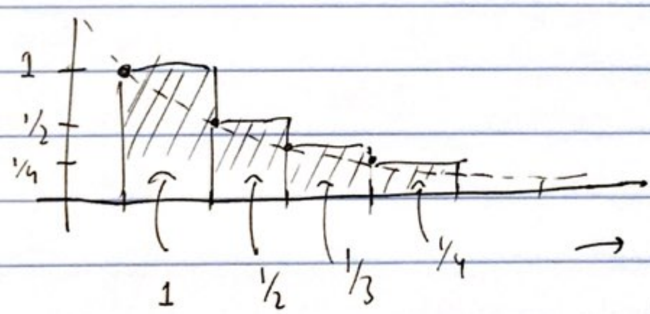
Let $a_n = f(n)$.

Then $\sum_{n=1}^{\infty} a_n$ converges if and only if.

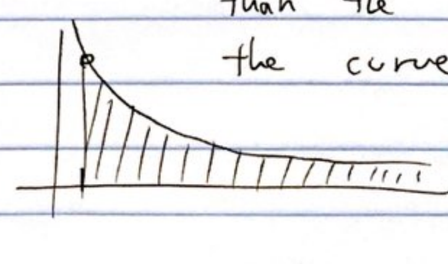
$\int_1^{\infty} f(x)\, dx$ is convergent.

Ex: $a_n = \frac{1}{n}$.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$



→ the sum is bigger than the area under the curve!



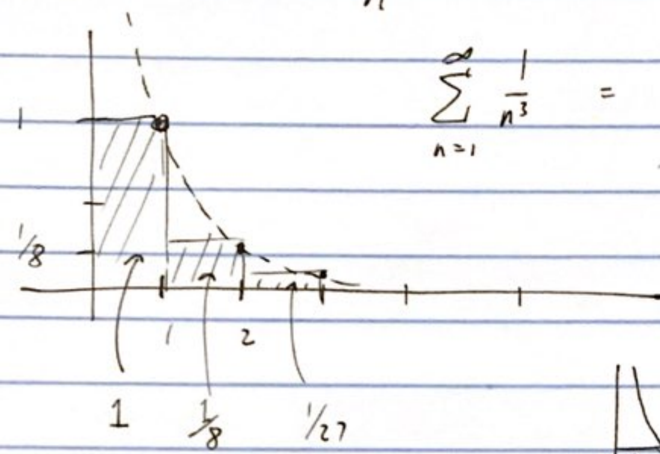
$$\int_1^{\infty} \frac{1}{x}\, dx$$

$$= \lim_{t \to \infty} \int_1^{t} \frac{1}{x}\, dx$$

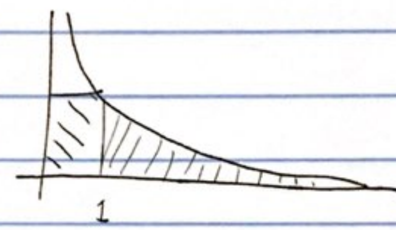
$$= \lim_{t \to \infty} \Big[\ln x\Big]_1^t = \lim_{t \to \infty} \left(\ln t - \ln 1\right) = \lim_{t \to \infty} \ln t = \infty$$

So $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

EX $a_n = \frac{1}{n^3}$



$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \dots$$



the area under the curve is bigger than the sum (after we include the first rectangle)

$$\int_1^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^3}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{x^{-2}}{-2} \right]_1^t$$

$$= \lim_{t\to\infty} \left(\frac{t^{-2}}{-2} - \frac{1}{-2} \right)$$

$$= \lim_{t\to\infty} \left(\frac{-1}{2t^2} + \frac{1}{2} \right) = 0 + \frac{1}{2} = \frac{1}{2}$$

So, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges. We dont know yet to what, since its just smaller than $\frac{3}{2}$.

Actually comes to 1.2020569....

p-series.

In fact, for any value p

$$\sum_{i=1}^{\infty} \frac{1}{n^p}$$

converges if $p > 1$

diverges if $p \le 1$.