Finding limits:

If $\lim\limits_{x \to \infty} f(x) = L$ (for $x \in \mathbb{R}$)

and $a_n = f(n)$ then $\lim\limits_{n \to \infty} a_n = L$.

... so we can use all the limits at ∞ from calc 1.

Ex: Find $\lim\limits_{n \to \infty} \dfrac{e^{2n} + n}{3e^{2n} - 5} = \dfrac{\infty}{\infty}$

$$= \lim_{x \to \infty} \frac{e^{2x} + x}{3e^{2x} - 5}$$

L'Hospitals

$$= \lim_{x \to \infty} \frac{2e^{2x} + 1}{6e^{2x}} = \frac{\infty}{\infty}$$

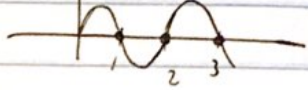
$$= \lim_{x \to \infty} \frac{4e^{2x}}{12e^{2x}}$$

$$= \lim_{x \to \infty} \frac{4}{12} = \boxed{\frac{1}{3}}$$

Ex: $\lim\limits_{n \to \infty} \sin(\pi n)$

$$= \boxed{0}$$

n	a_n
1	0
2	0
3	0
4	0
5	0



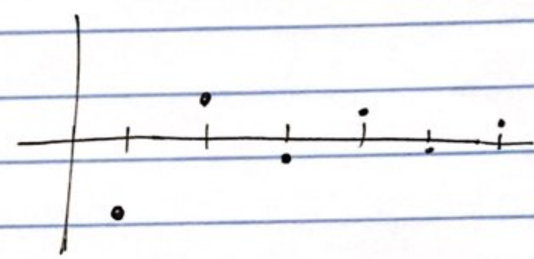
even though $\lim\limits_{x \to \infty} \sin(\pi x) = \text{DNE}$.

Theorem

If $\lim\limits_{n\to\infty} |a_n| = 0$ then $\lim\limits_{n\to\infty} a_n = 0$

Ex: $\lim\limits_{n\to\infty} \frac{(-1)^n}{2n^2}$

n	a_n
1	$-1/2$
2	$1/8$
3	$-1/18$
4	$1/32$
5	
6	



Note $|a_n| = \left|\frac{(-1)^n}{2n^2}\right| = \frac{|(-1)^n|}{|2n^2|} = \frac{1}{2n^2}$

and $\lim\limits_{n\to\infty} \frac{1}{2n^2} = 0$

so $\boxed{\lim\limits_{n\to\infty} \frac{(-1)^n}{2n^2} = 0}$

Doesn't work unless limit is zero!
(see $\lim\limits_{n\to\infty} (-1)^n = DNE$)

Ex $\lim\limits_{n\to\infty} e^{\frac{1}{n}} = \lim\limits_{x\to\infty} e^{1/x} = e^0 = 1$

11.2

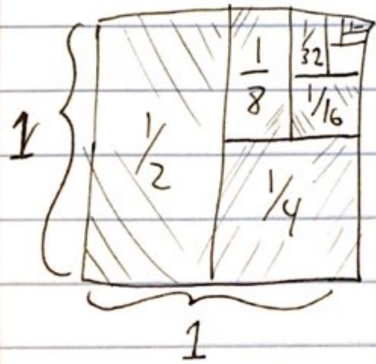
Summing a sequence: we call the sum of a sequence an <u>infinite</u> <u>series</u>

Ex: $$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

We want to know: will it add up to ∞ or to a real number?

Ex: $$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

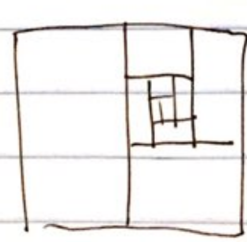
This infinite sum actally gets closer and closer to adding up to a real #.



gets closer and closer to filling the box!

so $$\lim_{n \to \infty} \left(\sum_{i=1}^{n} \frac{1}{2^i} \right) = 1$$

alt picture



n	$\sum_{i=1}^{n} \frac{1}{2i}$
1	$\frac{1}{2}$
2	$\frac{3}{4}$
3	$\frac{7}{8}$

We have seen this before:

$$2 + \frac{7}{10} + \frac{1}{100} + \frac{8}{1000} + \frac{2}{10000} + \frac{8}{100000} + \frac{2}{1000000} + \dots$$

$$= 2.718282 \dots$$

$$= e$$

Any decimal number is an infinite sum of fractions: if its irrational then all the numerators are used, or if its repeating -

$$0.3333\overline{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \dots$$

$$= \frac{1}{3}$$

If the series sums to a real number, we say its convergent. If not, divergent. We often write that a convergent series "=" its limit.

Ex: geometric series

$$\sum_{n=1}^{\infty} (r)^n = \frac{r}{1-r} \quad \text{for fractions } |r| < 1$$

ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1.$

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{3}{10^n} = 3 \sum_{n=1}^{\infty} \left(\frac{1}{10^n}\right)$

$$= 3 \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$$

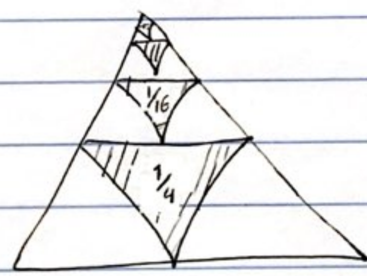
$$= 3 \frac{\frac{1}{10}}{1-\frac{1}{10}}$$

$$= 3 \frac{\frac{1}{10}}{\frac{9}{10}}$$

$$= 3\left(\frac{1}{10} \cdot \frac{10}{9}\right) = 3\left(\frac{1}{9}\right) = \frac{1}{3}.$$

Ex: $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{4} \cdot \frac{4}{3} = \frac{1}{3}$

$= \frac{1}{4} + \frac{1}{16} + \frac{1}{64}$



$= \frac{1}{3}$

Ex: $\displaystyle\sum_{n=1}^{\infty} 2^n = 2 + 4 + 8 + 16 + \ldots$

$$= \infty$$

Ex: $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n =$

$$= 1 + \frac{1}{4} + \frac{1}{16}$$

$$= 1 + \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$$

$$= 1 + \frac{1}{3} = \frac{4}{3}$$

$$= \frac{1}{1-\frac{1}{4}}$$

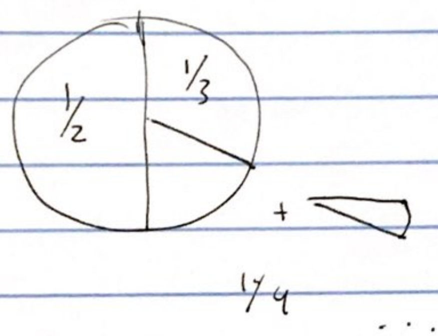
Whenever $\displaystyle\lim_{n\to\infty} a_n \neq 0$,

then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges

(Converse not true: lots of times $\lim a_n = 0$, but $\sum a_n$ still diverges.

Ex :
$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

$$= \infty$$



Proof: 11.3 Integral test.