Overview of integration $\int f(x)\,dx$

Methods: 1) reverse power rule

$$\int x^n dx = \frac{x^{n+1}}{n+1}, \quad n \neq 1$$

2) logs, exp. $\int \frac{1}{x} dx = \ln|x|$

$$\int e^x dx = e^x \qquad \int 5^x dx = \frac{5^x}{\ln 5}$$

3) trig $\int \cos x\,dx = \sin x$

(reverse all trig derivatives)

$f(x)$	$\int f(x)\,dx$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$\csc x \cot x$	$-\csc x$
$\csc^2 x$	$-\cot x$

4) u-substitution $\int e^{2x} dx = \frac{1}{2} e^{2x}$

$$\int \cos 7x\,dx = \frac{1}{7} \sin 7x$$

$$\int \tan x\,dx = \ln|\sec x|$$

$$\int \cot x\,dx = \ln|\sin x|$$

5) inverse trig

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\left(\frac{d}{dx}\ \frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{1}{a}\right.$$

Note: no need for $\cos^{-1}x$.

6) hyperbolic trig

$$\int \sinh x\,dx = \cosh x$$

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7) trig substitution.

$$\int \sqrt{a^2-x^2}\,dx \qquad x = a\sin\theta$$

$$\int \sqrt{a^2+x^2}\,dx \qquad x = a\tan\theta$$

$$\int \sqrt{x^2-a^2}\,dx \qquad x = a\sec\theta$$

8) int. by parts

$$\int u\,dv = uv - \int v\,du$$

9) Simplify: break up, long divide, partial fractions.

$$\int \overbrace{\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}}^{f(x)} dx$$

$$\begin{array}{r} x + 1 \\ x^3 - x^2 - x + 1 \;\big)\; x^4 - 2x^2 + 4x + 1 \\ -(x^4 - x^3 - x^2 + x) \\ \hline x^3 - x^2 + 3x + 1 \\ -(x^3 - x^2 - x + 1) \\ \hline 4x \end{array} \qquad R\ 4x$$

so $f(x) = x + 1 + \dfrac{4x}{x^3 - x^2 - x + 1}$

$$\frac{4x}{x^3 - x^2 - x + 1} = \frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

$$= A(x^2 - 1) + B(x+1) + C(x^2 - 2x + 1)$$

$$= Ax^2 - A + Bx + B + Cx^2 - 2Cx + C$$

$$= (A + C)x^2 + (B - 2C)x + (-A + B + C)$$

$\Rightarrow$ ① $A + C = 0 \rightsquigarrow A = -C$

② $B - 2C = 4$

③ $-A + B + C = 0 \longrightarrow C + B + C = 0$

$$\begin{array}{r} B - 2C = 4 \\ + \;\; B + 2C = 0 \\ \hline 2B = 4 \Rightarrow B = 2 \end{array}$$

② $2 - 2C = 4$

$-2C = 2$

$C = -1$

$A = 1$

Finally

$$\int f(x)\,dx = \int x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}$$

$$= \frac{x^2}{2} + x + \ln|x-1| + \frac{-2}{x-1} - \ln|x+1|$$